Conservation laws of helical flows

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Abstract

Helically symmetric flows are present in many technical devices such as wind turbine, combustion chamber and marine propeller. The understanding of the nature of this type of flow can be useful to describe such phenomena like a vortex breakdown. The aim of this thesis is the analytical investigation of helically symmetric flows in terms of conservation laws. After finding the conservation laws one is able to use the obtained theoretical results in order to construct exact solutions of the analyzed system of equations, furthermore the obtained divergence expressions are useful in numerics.

The introduction of a helical variable $\xi = az + b\varphi$, which is a twist of two cylindrical variables z and φ, allows to consider not only the helically symmetric system of equations, but also the two important limiting cases: the case of a plane flow and the case of rotationally symmetric flow. The theoretical procedure to find the conservation laws is called the direct construction method and is described in Anco, Bluman & Cheviakov (2010). This method is based on two ideas: application of the Euler operator and finding multipliers for the given system of equations. A Maple based package called GeM is used for the calculation of the local conservation laws. In order to obtain a global conservation law one has to integrate the local conserved quantities. For this, direct numerical simulations with a code HELIX (Delbende, Rossi & Daube 2012), which describes the dynamics of helically symmetric flows, were performed. By the introduction of helical symmetry the three-dimensional Navier-Stokes equations can be reduced to a two-dimensional problem. This numerical method is a generalization of the vorticity/stream function formulation in a circular domain, with finite differences in the radial direction and spectral decomposition along the azimuth. Compared to a standard three-dimensional code, this allows to reach large Reynolds numbers in feasible time.

For the analytical part of this work time dependent Euler and Navier-Stokes equations written in three different formulations are considered: in primitive variables, in stream function formulation and in vorticity formulation. Various new sets of conservation laws for both inviscid and viscous flows, including families that involve arbitrary functions, are derived. In particular, for inviscid flows, a family of conserved quantities, that generalize helicity, is obtained.

The special case of two-component flows, with zero velocity component in the invariant direction, is additionally considered, and special conserved quantities that hold for such flows are computed. In particular, it is shown that the well-known infinite set of generalized enstrophy conservation laws that holds for plane flows also holds for the general two-component helically invariant flows and for axisymmetric two-component flows.

For the integration of local conservation laws the time dependent Navier-Stokes and

Euler equations with three velocity components are considered. Using the integration the global conservation could be investigated. The remaining set of conservation laws for two-component flows could not be performed numerically due to the fundamental equations of the code, which include all velocity components.

It should be noted that the analytical part of this dissertation were published in Kelbin, Cheviakov & Oberlack (2013).

Kurzfassung

Strömungen, welche eine helikale Symmetrie besitzen, treten in vielen technischen Anwendungen auf, beispielsweise im Nachlauf von Schiffsschrauben oder Windturbinenschaufeln, oder in Brennkammern. Das fundamentale Verständnis dieser Art von Strömung kann nützlich sein um Phänomene wie beispielsweise den "vortex breakdown" zu beschreiben. Das Ziel der vorliegenden Arbeit ist die analytische Untersuchung der Strömungen mit helikaler Symmetrie im Hinblick auf Erhaltungsgleichungen. Nachdem die Erhaltungsgleichungen konstruiert wurden, kann man diese verwenden um beispielsweise exakte Lösungen des Ausgangsproblems zu erhalten, weiterhin sind die gefundenen Divergenzformulierungen nützlich für die numerische Berechnung.

Die Einführung der helikalen Variable ξ, welche eine Kombination aus zwei zylindrischen Koordinaten (z und φ) ist, ermöglicht nicht nur helikale Strömungen, sondern auch ebene und rotationssymmetrische Strömungen zu betrachten. Die Grundlagen des Verfahrens, welches für die Auffindung der Erhaltungsgleichungen benutzt wurde, sind in Anco et al. (2010) beschrieben. Diese Methode basiert auf zwei Ideen: die Anwendung des Euleroperators und das Bestimmen der Multiplikatoren für ein gegebenes Gleichungssystem. Die Berechnung von lokalen Erhaltungsgrößen wird mit dem Unterprogramm GeM realisiert, welches in das Softwareprogramm Maple eingebunden wird. Um aus der lokalen Formulierung der Erhaltungssätze die globalen Erhaltungsgleichungen zu bekommen, müssen die lokalen Erhaltungsgrößen integriert werden. Die Integration wurde mittels direkter numerischer Simulation durchgeführt, mit einem Code HELIX (Delbende et al. 2012), welcher für die Stabilitätsuntersuchungen von Strömungen mit helikaler Symmetrie programmiert wurde. Durch die Annahme der helikalen Symmetrie können die dreidimensionalen Navier-Stokes Gleichungen zu einem zweidimensionalen Gleichungssystem reduziert werden. Die dem Code zugrundeliegende numerische Methode ist eine Verallgemeinerung der Wirbelstärke-Stromfunktion-Formulierung. In radialer Richtung wird die Finite-Differenzen-Methode benutzt, in azimutaler Richtung die Spektralzerlegung. Verglichen mit einem herkömmlichen Computerprogramm für dreidimensionale Probleme können höhere Reynoldszahlen erreicht werden.

In dem analytischen Teil der vorliegenden Arbeit (Bestimmung der lokalen Erhaltungsgleichungen) hat man die instationären Euler- beziehungsweise Navier-Stokes Gleichungen in drei verschiedenen Formulierungen untersucht: in primitiven Variablen, in Stromfunktionformulierung und in Wirbelstärkeformulierung. Verschiedene neue Erhaltungsgleichungen, sowohl für reibungsfreie als auch für reibungsbehaftete Strömungen, konnten hergeleitet werden. Insbesondere für reibungsfreie Strömungen konnte man eine Familie von Erhaltungsgleichungen herleiten, welche die Helizität verallgemeinern.

Der Spezialfall von Zweikomponentenströmung, bei der die Geschwindigkeitskomponente in die invariante Richtung zu Null gesetzt wird, ist ebenfalls betrachtet worden. Für diesen Fall wurden weitere neue Erhaltungsgleichungen konstruiert. Insbesondere konnte gezeigt werden, dass die für ebene Strömungen wohlbekannte Erhaltungsgleichungfamilie der verallgemeinerten Enstrophie auch im Fall der zweidimensionalen helikalen Strömung ihre Gültigkeit besitzt.
Für die numerische Integration der lokalen Erhaltungsgleichungen wurden nur instationären Euler- und Navier-Stokes Gleichungen mit drei Geschwindigkeitskomponenten betrachtet. Die globalen Erhaltungseigenschaften konnten somit untersucht werden. Die übrigen Erhaltungsgrößen (Zweikomponentenströmungen) konnten nicht betrachtet werden, da die dem Code zugrundeliegenden Gleichungen alle drei Geschwindigkeitskomponenten benötigen.

Es wird darauf hingewiesen, dass die analytischen Ergebnisse dieser Arbeit in Kelbin et al. (2013) veröffentlicht wurden.

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1 Introduction

Flows that exhibit helically symmetric behaviour appear in a wide range of natural phenomena and fluid mechanics applications. Some basic examples include helical vortex structures that arise as unstable modes as a result of vortex breakdown in swirling jets (Sarpkaya 1971). Helical vortices are experimentally observed in various technological devices with swirling, in particular, cyclones (Gupta & Kumar 2007) and tubular burners (Ishizuka 1989), in the wake of windmills (Vermeer, Sorensen & Crespo 2003) or as wing tip vortices, in particular, on delta wings (Mitchell, Morton & Forsythe 1997). A number of different helical vortex structures emerge in vortex chambers under different boundary conditions and have been described by Alekseenko, Kuibin, Okulov & Shtork (1999). Experiments involving viscous liquid jets discharged from a long vertical rotating tube demonstrating fast development of helical flow downstream of the rotating tube are described by Kubitschek & Weidman (2008). Interestingly, helical instabilities in swirl flows appear not only in laminar, but also in turbulent flows. For example, double helical structures have been observed in a number of settings (Chandrsuda, Mehta, Weir & Bradshaw 1978, Alekseenko et al. 1999). A review of swirling flows with helical structure in technical applications is given by Alekseenko & Okulov (1996).

For rotating pipe flows, stable helical waves analogous to the two-dimensional nonlinear waves in plane Poiseuille flows have been observed in numerical simulations by Toplosky & Akylas (1988), while time-dependent helical waves for the full Navier-Stokes equations in rotating pipe flow were computed by Landman (1990b). Similar helical structures are also known to arise in stationary pipes with swirl in the inlet flow (Landman 1990a).

Helically symmetric flows and equilibrium configurations are also of interest in magnetohydrodynamics (Dritschel 1991). In plasma physics they naturally arise both in laboratory plasma applications (kink instabilities in the "straight tokamak" approximations (Schnack, Caramana & Nebel 1985, Johnson, Oberman, Kulsrud & Frieman 1958)) and astrophysical phenomena such as astrophysical jets (Bogoyavlenskij 2000).

In the last few decades, various authors have contributed to the theoretical description of helical flows. In the most straightforward approach, the helical symmetry is imposed by assuming the spatial dependence of all physical variables on the cylindrical radius r and the helical variable $\xi = az + b\varphi$ $(a, b = const. \neq 0)$. In this ansatz, both the system of static plasma equilibrium equations and the system of steady Euler equations of incompressible fluid dynamics collapse to a single equation - the well-known JFKO equation (Johnson et al. 1958). In a more general setting, twisted pipes following a given spatial curve have been considered in a number of works (Wang 1981, Germano 1982, Germano 1989, Tuttle 1990). In particular, effects of pipe curvature and torsion on the flow were studied using suitable (non-orthogonal and locally orthogonal) coordinate systems. Analytical solutions describing helical flows have appeared in a number of works although they have emerged from different settings. In particular, steady flow solutions in helically symmetric pipes were obtained by Zabielski & Mestel (1998). Helical static plasma equilibria modelling isotropic and anisotropic astrophysical jets were derived in Bogoyavlenskij (2000), Cheviakov & Bogoyavlenskij (2004).

The question of existence and uniqueness of time-dependent helically invariant inviscid flows was addressed by Ettinger & Titi (2009), where uniqueness and existence of weak solutions of Euler equations were proved under the physical geometric constraint of no vorticity stretching. This, as will be seen subsequently, is a consequence of a zero velocity component in the invariant direction. In contrast, existence and uniqueness of the helically symmetric Navier-Stokes equations without further constraints were proven by Mahalov, Titi & Leibovich (1990).

The main goals of this dissertation are the derivation and analysis of the full three-dimensional system of incompressible constant-density Euler and Navier-Stokes equations under the assumption of helical symmetry, and, in particular, the derivation of the conservation laws admitted by this system. In the general helically symmetric setting, all three velocity components and pressure are generally nonzero. They depend on time t, and, employing a cylindrical coordinate system, the cylindrical radius r and the helical variable

$$\xi = az + b\varphi.$$

The considered helically symmetrical setting is thus purely based on the independence on the third spatial variable (measured along each helix), and no restrictive assumptions whatsoever are made about the form of velocity components or pressure. The flow therefore has two spatial dimensions and is naturally referred to as $(2+1)$-dimensional in space-time. Since independent space dimensions are reduced to two and the flow has three independent components of the velocity vector, it is often referred to as $2\frac{1}{2}$-dimensional flow. As commonly accepted in turbulence research, a flow is referred to as a two-component flow when one of the velocity components is set to zero.

Helical coordinates that employ the above form of the helical variable ξ provide a natural transition between cartesian and cylindrical coordinates, and lets one impose helical invariance, which generalizes both the axial symmetry (achieved at $a = 1, b = 0$) and the translational symmetry ($a = 0, b = 1$). In the following chapter 2, the general helically symmetric Navier-Stokes equations in the primitive variables as well as in the vorticity formulation will be derived. These formulae generalize the helically invariant inviscid model discussed by Alekseenko et al. (1999). Important special cases of planar and axially symmetric flows in a helically symmetric setting are also analysed. The vorticity formulation is employed to derive multiple additional conservation laws of the helically invariant Euler and Navier-Stokes equations.

A helically symmetric stream function formulation of the Euler and Navier-Stokes equations will also be derived in a straightforward manner. However, this formulation will not be explicitly considered in conservation laws analysis since it yields no additional conservation laws compared with those obtained from primitive or vorticity variables.

In this dissertation local conservation laws for helically invariant Navier-Stokes and Euler equations will be systematically constructed. A local conservation law is a divergence expression

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{\Phi} = 0, \tag{1.1}$$

where Θ is the density and components of $\mathbf{\Phi}$ are spatial fluxes. In particular, if the original equations include viscous terms, such terms must be included into the divergence expression.
Conservation laws (1.1) have multiple applications. In particular, it follows from the Gauss theorem that if the fluxes $\mathbf{\Phi}$ vanish on the boundary of the fluid domain D or at infinity, each conservation law (1.1) yields a globally conserved quantity

$$R = \iiint_D \Theta \; dV, \quad \frac{\partial R}{\partial t} = 0. \tag{1.2}$$

Moreover, the knowledge of local conservation laws (1.1) admitted by systems of fluid dynamics equations is important from the point of view of numerical modeling. Indeed, multiple modern finite-element methods, such as discontinuous Galerkin methods, are based on divergence forms of the given equations. Conservation laws are also useful in partial differential equation (PDE) analysis, in particular, studies of existence, uniqueness and stability of solutions of nonlinear PDEs, as well as for the construction of linearizations and exact solutions through nonlocally related PDE systems (e.g., Lax (1968), Benjamin (1972), Knops & Stuart (1984), Anco, Bluman & Wolf (2008), Bluman, Cheviakov & Ganghofer (2008)).

An important class of conservation laws in fluid dynamics are the material conservation laws given by vanishing material derivatives

$$\frac{d\Theta}{dt} \equiv \frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta = 0, \tag{1.3}$$

where $\boldsymbol{u}$ is a flow velocity vector. If (1.3) holds, the total amount of the quantity Θ initially assigned to any fluid parcel is conserved. For incompressible flows where $\nabla \cdot \boldsymbol{u} = 0$, each material conservation law (1.3) is equivalent to a local conservation law (1.1) with $\mathbf{\Phi} = \boldsymbol{u}\Theta$. A classical example of material conservation laws is the well-known family of vorticity conservation laws for plane flows (Bowman (2009); see also chapter 6.1 and formula (6.16) below). However, it is clear that not every conservation law (1.1) is equivalent to some material conservation law. In the papers by Moiseev, Sagdeev, Tur & Yanovskii (1982), Tur (1993), Volkov, Tur & Yanovsky (1995) and references therein, material conservation laws (1.3) are referred to as Lagrange invariants, and other types of invariants are considered for various hydrodynamic settings. Moiseev et al. (1982) used invariants to construct exact vortex-like solutions of a two-fluid hydrodynamics model.

The actual algorithmic construction of local conservation laws for complex models became feasible with an introduction of the direct construction method (Anco & Bluman 2002a, Anco & Bluman 2002b, Anco et al. 2010). The method is briefly reviewed in chapter 3. It stems from ideas related to Noether's theorem but is free from restrictive assumptions related to the existence of a variational formulation. The direct construction method is directly applicable to the vast majority of physical models.

Well-known classical conservation laws of three-dimensional time-dependent inviscid fluid dynamics include the conservation of mass, momentum, angular momentum, energy, vorticity, helicity and the so-called center-of-mass theorem (see, e.g., Batchelor (2000), Caviglia & Morro (1989), Moffatt (1969)). Chapter 4 is concerned with finding additional conservation laws of the helically invariant Euler system, both in primitive variables and in vorticity formulation. For helical flows, the above list can be substantially extended: helical Euler equations are shown to admit infinite sets of generalized momentum/angular momentum conservation laws and families of new vorticity conservation laws involving arbitrary functions. In particular, one such family corresponds to conservation of generalized helicity-type expressions.

In chapter 5, conservation laws of helically symmetric Navier-Stokes equations are studied. Owing to the essentially dissipative structure of the Navier-Stokes model, one might not expect to find many conservation laws for it. However, it is shown in chapter 5 that the helically symmetric Navier-Stokes dynamics conserves one component of the momentum, one component of the angular momentum, and an infinite number of additional vorticity-dependent expressions.

In chapter 6, the special case of two-component helically invariant inviscid flows is studied. For such flows, the velocity component in the invariant direction, u^η, is identically zero. As is well known, inviscid plane flows possess an infinite number of vorticity-related conservation laws, one of them being the enstrophy ω^2. Often they are referred to as Casimirs (Bowman 2009). This result could be generalized onto helically symmetric inviscid flows with vanishing velocity component u^η in the invariant direction. Moreover, several new sets of conservation laws are found for the specific cases of plane and axisymmetric flows with vanishing transverse velocity components, in both viscous and inviscid settings, in primitive and vorticity variables (chapter 6.2, 6.3). Some of these new sets generalize previously known results, whereas other conservation laws are new.

In chapter 7 the numerical code HELIX is described. The code was developed by Ivan Delbende, Maurice Rossi and Olivier Daube and is a property of Centre National de la Recherche Scientifique (CNRS). This DNS code will be used to integrate the local conservation laws of Euler and Navier-Stokes equations from chapters 4 and 5 to obtain the global conserved quantities. Two different types of initial conditions will be used. The results are outlined in section 7.3. Iso contour plots of the vorticity component ω^η in a plane at $z = 0$ are shown in the appendix A to get an information about the time evolution.

2 Helically invariant Navier-Stokes equations

2.1 Helical coordinates

The first step towards a solution of a problem is the introduction of an appropriate coordinate system. As an initial point we have the cylindrical coordinates r, φ and z, where r is the radial, φ the azimuthal and z the streamwise direction. In order to describe the helical motion one introduces a helical variable ξ which is defined as:

$$\xi = az + b\varphi,$$

and can be understood as a combination of two cylindrical coordinates φ and z. The second helical variable η is defined as:

$$\eta = a\varphi - bz/r^2.$$

On each cylinder $r = const.$, lines of $\xi = const.$ and $\eta = const.$ correspond to two families of helices on that cylinder. The constants a and b have only to fulfill the condition $a^2 + b^2 > 0$. The choice of the constants a, b prescribes a specific helical frame. There exist three different interesting settings for a and b, which will define different type of geometries of the flow. The case $a = 1, b = -\frac{h}{2\pi}$ corresponds to the helically symmetric flow (see figure 2.1). In the limiting case $a = 1, b = 0$, helical coordinates become cylindrical coordinates with $\eta = \varphi, \xi = z$ and in the case $a = 0, b = 1$ one obtains $\eta = z, \xi = \varphi$ which can be understood as a plane flow.
Any helically invariant function of time and spatial variables is a function independent of η, and has the form $F(t, r, \xi)$. Since the goal of this thesis is to examine helically symmetric flows, the physical variables will be assumed $\eta-$independent. It is worth noting that in the limiting case $a = 1, b = 0$, the helical symmetry reduces to the axial symmetry; in the opposite case $a = 0, b = 1$, the helical symmetry corresponds to the planar symmetry, i.e., symmetry with respect to translation in the z-direction.

It should be noted that helical coordinates by (r, η, ξ) are not orthogonal. In fact, although the coordinates r, ξ are orthogonal, there exists no third coordinate orthogonal to both r and ξ. To see this, one has to derive the basis unit vectors via formula

$$\boldsymbol{e}_i = \frac{\nabla i}{|\nabla i|} \tag{2.1}$$

using the helical variables:

$$r, \quad \eta = a\varphi - bz/r^2, \quad \xi = az + b\varphi. \tag{2.2}$$

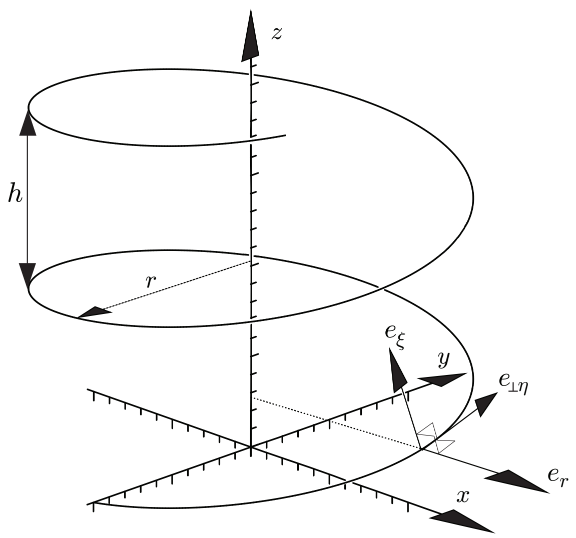


Figure 2.1: An illustration of the helix $\xi = const.$, h is the $z-$step over one helical turn.

First one has to calculate the gradients of the coordinates. A gradient of a scalar S written in cylindrical coordinates is equal to:

$$\mathrm{grad}S = \nabla S = \frac{\partial S}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial S}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial S}{\partial z}\boldsymbol{e}_z.$$

For the radius r it leads to:

$$\nabla r = \frac{\partial r}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial r}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial r}{\partial z}\boldsymbol{e}_z = 1\boldsymbol{e}_r + 0\boldsymbol{e}_\varphi - 0\boldsymbol{e}_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|\nabla r| = \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| = \sqrt{1} = 1.$$

In case of the helical coordinate η one obtains:

$$\begin{aligned}\nabla \eta = \frac{\partial \eta}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \eta}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial \eta}{\partial z}\boldsymbol{e}_z =& \frac{\partial\left(a\varphi - bz/r^2\right)}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\left(a\varphi - bz/r^2\right)}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial\left(a\varphi - bz/r^2\right)}{\partial z}\boldsymbol{e}_z \\ =& \frac{2bz}{r^3}\boldsymbol{e}_r + \frac{1}{r}a\boldsymbol{e}_\varphi - \frac{b}{r^2}\boldsymbol{e}_z = \begin{pmatrix} \frac{2bz}{r^3} \\ \frac{a}{r} \\ -\frac{b}{r^2} \end{pmatrix},\end{aligned}$$

$$\begin{aligned}|\nabla \eta| =& \left| \begin{pmatrix} \frac{2bz}{r^3} \\ \frac{a}{r} \\ -\frac{b}{r^2} \end{pmatrix} \right| = \sqrt{\left(\frac{2bz}{r^3}\right)^2 + \left(\frac{a}{r}\right)^2 + \left(-\frac{b}{r^2}\right)^2} = \sqrt{\frac{1}{r^2}\left(\frac{4b^2z^2}{r^4} + a^2 + \frac{b^2}{r^2}\right)} \\ =& \sqrt{\frac{1}{r^2}\left(\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}\right)} = \frac{1}{r}\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}},\end{aligned}$$

where $B(r)$ is a parameter function defined by:

$$B(r) = \frac{r}{\sqrt{a^2r^2+b^2}}.$$

For the third coordinate ξ one has:

$$\nabla\xi = \frac{\partial\xi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\xi}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial\xi}{\partial z}\boldsymbol{e}_z = \frac{\partial\left(az+b\varphi\right)}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\left(az+b\varphi\right)}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial\left(az+b\varphi\right)}{\partial z}\boldsymbol{e}_z$$
$$= 0\boldsymbol{e}_r + \frac{1}{r}b\boldsymbol{e}_\varphi + a\boldsymbol{e}_z = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix},$$

$$|\nabla\xi| = \left|\begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix}\right| = \sqrt{0^2 + \left(\frac{b}{r}\right)^2 + a^2} = \frac{1}{B(r)}.$$

The basis unit vectors obtained by this procedure are:

$$\boldsymbol{e}_r = \frac{\nabla r}{|\nabla r|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

$$\boldsymbol{e}_\eta = \frac{\nabla_\eta}{|\nabla_\eta|} = \begin{pmatrix} \frac{2bz}{r^3} \\ \frac{a}{r} \\ -\frac{b}{r^2} \end{pmatrix} \frac{r}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}} = \begin{pmatrix} \frac{2bz}{r^2} \\ a \\ -\frac{b}{r} \end{pmatrix} \frac{1}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}}, \tag{2.4}$$

and

$$\boldsymbol{e}_\xi = \frac{\nabla\xi}{|\nabla\xi|} = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r). \tag{2.5}$$

The basis unit vectors have to fulfill the orthogonality conditions:

$$\boldsymbol{e}_r \times \boldsymbol{e}_\eta = \boldsymbol{e}_\xi, \quad \boldsymbol{e}_\eta \times \boldsymbol{e}_\xi = \boldsymbol{e}_r, \quad \boldsymbol{e}_\xi \times \boldsymbol{e}_r = \boldsymbol{e}_\eta. \tag{2.6}$$

The first identity reads:

$$\boldsymbol{e}_r \times \boldsymbol{e}_\eta \stackrel{!}{=} \boldsymbol{e}_\xi$$

$$\boldsymbol{e}_r \times \boldsymbol{e}_\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{2bz}{r^2} \\ a \\ -\frac{b}{r} \end{pmatrix} \frac{1}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}} = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} \frac{1}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}} \neq \boldsymbol{e}_\xi = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r).$$

The second identity

$$\boldsymbol{e}_\eta \times \boldsymbol{e}_\xi \stackrel{!}{=} \boldsymbol{e}_r$$

leads to:

$$\boldsymbol{e}_\eta \times \boldsymbol{e}_\xi = \begin{pmatrix} \frac{2bz}{r^2} \\ a \\ -\frac{b}{r} \end{pmatrix} \frac{1}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}} \times \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) = \begin{pmatrix} a^2 + \frac{b^2}{r^2} \\ -\frac{2abz}{r^2} \\ -\frac{2b^2z}{r^3} \end{pmatrix} \frac{B(r)}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}} \neq \boldsymbol{e}_r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, the last identity:

$$\boldsymbol{e}_\xi \times \boldsymbol{e}_r \overset{!}{=} \boldsymbol{e}_\eta$$

$$\boldsymbol{e}_\xi \times \boldsymbol{e}_r = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ -\frac{b}{r} \end{pmatrix} B(r) \neq \boldsymbol{e}_\eta = \begin{pmatrix} \frac{2bz}{r^2} \\ a \\ -\frac{b}{r} \end{pmatrix} \frac{1}{\sqrt{\frac{4b^2z^2}{r^4} + \frac{1}{B(r)^2}}}.$$

However, an orthogonal basis is readily constructed at any point except for the origin, as follows (see Figure 2.1):

$$\boldsymbol{e}_r = \frac{\nabla r}{|\nabla r|}, \quad \boldsymbol{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|},$$

$$\boldsymbol{e}_{\perp\eta} := \boldsymbol{e}_\xi \times \boldsymbol{e}_r = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ -\frac{b}{r} \end{pmatrix} B(r).$$

Now one can proof once again the orthogonality properties of the basis unit vectors:

$$\boldsymbol{e}_r \times \boldsymbol{e}_{\perp\eta} = \boldsymbol{e}_\xi, \quad \boldsymbol{e}_{\perp\eta} \times \boldsymbol{e}_\xi = \boldsymbol{e}_r, \quad \boldsymbol{e}_\xi \times \boldsymbol{e}_r = \boldsymbol{e}_{\perp\eta}.$$

One obtains:

$$\boldsymbol{e}_r \times \boldsymbol{e}_{\perp\eta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ a \\ -\frac{b}{r} \end{pmatrix} B(r) = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) = \boldsymbol{e}_\xi,$$

$$\boldsymbol{e}_{\perp\eta} \times \boldsymbol{e}_\xi = \begin{pmatrix} 0 \\ a \\ -\frac{b}{r} \end{pmatrix} B(r) \times \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) = \begin{pmatrix} a^2 + \frac{b^2}{r^2} \\ 0 \\ 0 \end{pmatrix} B(r)^2 = \begin{pmatrix} \frac{1}{B(r)^2} \\ 0 \\ 0 \end{pmatrix} B(r)^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{e}_r,$$

$$\boldsymbol{e}_\xi \times \boldsymbol{e}_r = \begin{pmatrix} 0 \\ \frac{b}{r} \\ a \end{pmatrix} B(r) \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ -\frac{b}{r} \end{pmatrix} B(r) = \boldsymbol{e}_{\perp\eta}.$$

The orthonormal basis is then defined by the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_{\perp\eta}$ and $\boldsymbol{e}_\xi$. By reason of readability of the equations one will keep the index $(\cdot)_\eta$ instead of $(\cdot)_{\perp\eta}$. Further, for brevity, one will write $B(r) = B$ and $dB(r)/dr = B'$.

In order to rewrite the Navier-Stokes equations in helical coordinate system, one needs to substitute the derivatives with respect to cylindrical coordinates by the derivatives with respect to helical coordinates (2.2):

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial}{\partial r} \\
\frac{\partial}{\partial \varphi} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial \varphi} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial \varphi} = 0 + a\frac{\partial}{\partial \eta} + b\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial z} = 0 - \frac{b}{r^2}\frac{\partial}{\partial \eta} + a\frac{\partial}{\partial \xi}
\end{aligned}
\tag{2.7}
$$

After all necessary preparations are made, we write the Navier-Stokes equations in helically symmetrical setting, which will be done in the next section.

2.2 Equations in primitive variables formulation

We consider the Navier-Stokes equations, which describe the motion of a fluid, at constant viscosity ν, without loss of generality we set external forces to zero and assume density being a constant ($\rho = const. = 1$). Then the equations are given by:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.8a}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \nu\, \nabla^2 \boldsymbol{u} = 0. \tag{2.8b}$$

where $\boldsymbol{u}$ is the velocity vector and p is pressure. The inviscid case $\nu = 0$ yields the Euler equations.
In cylindrical coordinate system the equations (2.8a) and (2.8b) are given by:

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + \frac{1}{r}\frac{\partial u^\varphi}{\partial \varphi} + \frac{\partial u^z}{\partial z} = 0, \tag{2.9a}$$

$$
\begin{aligned}
\frac{\partial u^r}{\partial t} + u^r\frac{\partial u^r}{\partial r} + \frac{1}{r}\left(u^\varphi\frac{\partial u^r}{\partial \varphi} - (u^\varphi)^2\right) + u^z\frac{\partial u^r}{\partial z} &= -\frac{\partial p}{\partial r} \\
&\quad + \nu\left[\Delta u^r - \frac{1}{r^2}\left(u^r + 2\frac{\partial u^\varphi}{\partial \varphi}\right)\right],
\end{aligned}
\tag{2.9b}
$$

$$
\begin{aligned}
\frac{\partial u^\varphi}{\partial t} + u^r\frac{\partial u^\varphi}{\partial r} + \frac{1}{r}\left(u^\varphi\frac{\partial u^\varphi}{\partial \varphi} + u^r u^\varphi\right) + u^z\frac{\partial u^\varphi}{\partial z} &= -\frac{\partial p}{\partial \varphi} \\
&\quad + \nu\left[\Delta u^\varphi - \frac{1}{r^2}\left(u^\varphi - 2\frac{\partial u^r}{\partial \varphi}\right)\right],
\end{aligned}
\tag{2.9c}
$$

$$\frac{\partial u^z}{\partial t} + u^r\frac{\partial u^z}{\partial r} + \frac{1}{r}u^\varphi\frac{\partial u^z}{\partial \varphi} + u^z\frac{\partial u^z}{\partial z} = -\frac{\partial p}{\partial z} + \nu\Delta u^z, \tag{2.9d}$$

where Δ denotes the Laplace operator defined by:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

In order to rewrite the set of equations (2.9) in a helically symmetric setting, one starts by writing the velocity vector in the helical basis:

$$\boldsymbol{u} = u^r \boldsymbol{e}_r + u^\varphi \boldsymbol{e}_\varphi + u^z \boldsymbol{e}_z = u^r \boldsymbol{e}_r + u^\eta \boldsymbol{e}_{\perp\eta} + u^\xi \boldsymbol{e}_\xi. \tag{2.10}$$

The helical velocity components are related to the cylindrical velocity components by:

$$u^\eta = \boldsymbol{u} \cdot \boldsymbol{e}_{\perp\eta} = B\left(a u^\varphi - \frac{b}{r} u^z\right), \qquad u^\xi = \boldsymbol{u} \cdot \boldsymbol{e}_\xi = B\left(\frac{b}{r} u^\varphi + a u^z\right). \tag{2.11}$$

The reverse relations are given by:

$$u^\varphi = B\left(a u^\eta + \frac{b}{r} u^\xi\right), \qquad u^z = B\left(-\frac{b}{r} u^\eta + a u^\xi\right). \tag{2.12}$$

Now one is able to write the continuity equation and the momentum equations (2.9) in helical coordinates. A step by step transformation on the example of the continuity equation (2.9a) is shown here and can be easily transferred to the remaining equations: one starts with the equation (2.9a) and replaces the velocity components u^φ and u^z through u^η and u^ξ via relation (2.12):

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{1}{r}\frac{\partial u^\varphi}{\partial \varphi} + \frac{\partial u^z}{\partial z} = 0$$

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi}\left\{B\left(a u^\eta + \frac{b}{r} u^\xi\right)\right\} + \frac{\partial}{\partial z}\left\{B\left(-\frac{b}{r} u^\eta + a u^\xi\right)\right\} = 0.$$

In the next step one replaces the derivatives $\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}$ by $\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \xi}$ using (2.7):

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{1}{r}\left(a\frac{\partial}{\partial \eta} + b\frac{\partial}{\partial \xi}\right)\left\{B\left(a u^\eta + \frac{b}{r} u^\xi\right)\right\} + \left(\frac{b}{r^2}\frac{\partial}{\partial \eta} + a\frac{\partial}{\partial \xi}\right)\left\{B\left(-\frac{b}{r} u^\eta + a u^\xi\right)\right\} = 0.$$

Finally, imposing helical invariance $\frac{\partial}{\partial \eta} \equiv 0$ leads to:

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{b}{r}\frac{\partial}{\partial \xi}\left\{B\left(a u^\eta + \frac{b}{r} u^\xi\right)\right\} + a\frac{\partial}{\partial \xi}\left\{B\left(-\frac{b}{r} u^\eta + a u^\xi\right)\right\} = 0.$$

This equation can be simplified by rearrangement of the terms and executing the derivatives:

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{bB}{r}\left(\frac{\partial}{\partial \xi}\left(a u^\eta\right) + \frac{\partial}{\partial \xi}\left(\frac{b}{r} u^\xi\right)\right) + aB\left(-\frac{\partial}{\partial \xi}\left(\frac{b}{r} u^\eta\right) + a\frac{\partial}{\partial \xi}\left(u^\xi\right)\right) = 0$$

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + \frac{bB}{r}\left(a\frac{\partial u^\eta}{\partial \xi} + \frac{b}{r}\frac{\partial u^\xi}{\partial \xi}\right) + aB\left(-\frac{b}{r}\frac{\partial u^\eta}{\partial \xi} + a\frac{\partial u^\xi}{\partial \xi}\right) = 0$$

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + \frac{abB}{r}\frac{\partial u^\eta}{\partial \xi} + \frac{b^2 B}{r^2}\frac{\partial u^\xi}{\partial \xi} - \frac{abB}{r}\frac{\partial u^\eta}{\partial \xi} + a^2 B\frac{\partial u^\xi}{\partial \xi} = 0$$

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + B\left(\frac{b^2}{r^2} + a^2\right)\frac{\partial u^\xi}{\partial \xi} = 0$$

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + B\left(\frac{b^2 + a^2 r^2}{r^2}\right)\frac{\partial u^\xi}{\partial \xi} = 0$$

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + B\frac{1}{B^2}\frac{\partial u^\xi}{\partial \xi} = 0$$

and finally one obtains the continuity equation in helically symmetric notation:

$$\frac{1}{r}u^r + \frac{\partial u^r}{\partial r} + \frac{1}{B}\frac{\partial u^\xi}{\partial \xi} = 0.$$

In the same manner one obtains the momentum equation (2.13b) for u^r: one starts with the r-momentum equation (2.9b) written in cylindrical coordinates, substitutes the velocities u^φ, u^z by u^η, u^ξ using the relations (2.12) and the derivatives $\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}$ with relations (2.7).

A slightly different procedure is used to obtain the momentum equations (2.13c) and (2.13d) for the velocities u^η and u^ξ respectively: one has to build a linear combination of the u^φ- and u^z-momentum equation (2.9c) and (2.9d):

$$\text{(2.13c)} = B \cdot \left(a \cdot \text{(2.9c)} - \frac{b}{r} \cdot \text{(2.9d)}\right)$$

$$\text{(2.13d)} = B \cdot \left(\frac{b}{r} \cdot \text{(2.9c)} + a \cdot \text{(2.9d)}\right)$$

Remark: for reasons of readability, from now upper indices will refer to the corresponding components of vector fields (vorticity, velocity, etc.), and lower indices will denote partial derivatives. For example,

$$(u^\eta)_\xi \equiv \frac{\partial}{\partial \xi}u^\eta(t, r, \xi).$$

One also assumes summation in all repeated indices.

The procedure, outlined above, yields the following four equations constituting the helically invariant Navier-Stokes system in primitive variables:

$$\frac{1}{r}u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0, \tag{2.13a}$$

$$(u^r)_t + u^r(u^r)_r + \frac{1}{B}u^\xi(u^r)_\xi - \frac{B^2}{r}\left(\frac{b}{r}u^\xi + au^\eta\right)^2 = -p_r$$
$$+\nu\left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2bB}{r^2}\left(a(u^\eta)_\xi + \frac{b}{r}(u^\xi)_\xi\right)\right], \quad (2.13b)$$

$$(u^\eta)_t + u^r(u^\eta)_r + \frac{1}{B}u^\xi(u^\eta)_\xi + \frac{a^2B^2}{r}u^r u^\eta$$
$$= \nu\left[\frac{1}{r}(r(u^\eta)_r)_r + \frac{1}{B^2}(u^\eta)_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}u^\eta + \frac{2abB}{r^2}\left((u^r)_\xi - \left(Bu^\xi\right)_r\right)\right], \quad (2.13c)$$

$$(u^\xi)_t + u^r(u^\xi)_r + \frac{1}{B}u^\xi(u^\xi)_\xi + \frac{2abB^2}{r^2}u^r u^\eta + \frac{b^2B^2}{r^3}u^r u^\xi = -\frac{1}{B}p_\xi$$
$$+\nu\left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4B^4-1}{r^2}u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(u^r)_\xi + \left(\frac{aB}{r}u^\eta\right)_r\right)\right], \quad (2.13d)$$

where the velocity components u^r, u^η, u^ξ and pressure p are functions of r, ξ and t.

2.2.1 Rotationally symmetric and axisymmetric flows

As already mentioned above, the general helically symmetric equations include such interesting types of flows as rotationally symmetric and plane flows. Within this work "rotationally symmetric flow" means a flow with all parameters independent of the polar angle φ and all three velocity components are nonzero. Equations governing such flows are obtained by setting

$$a = 1, \quad b = 0, \quad B = 1, \quad \xi = az + b\varphi = 1z + 0\varphi = z \quad (2.14)$$

in the equations (2.13). Observing that

$$u^\xi = B\left(\frac{b}{r}u^\varphi + au^z\right) = 1\,(0u^\varphi + 1u^z) = u^z$$

and

$$u^\eta = B\left(au^\varphi - \frac{b}{r}u^z\right) = 1\,(1u^\varphi - 0u^z) = u^\varphi,$$

one obtains a system of rotationally symmetric Navier-Stokes equations. A further reduction, referred to as "axisymmetric flow", corresponds to the absence of flow in the polar direction: $u^\varphi = 0$, and is given by:

$$\frac{1}{r}u^r + (u^r)_r + (u^z)_z = 0, \quad (2.15a)$$

$$(u^r)_t + u^r(u^r)_r + u^z(u^r)_z = -p_r + \nu\left[\frac{1}{r}(r(u^r)_r)_r + (u^r)_{zz} - \frac{1}{r^2}u^r\right], \quad (2.15b)$$

$$(u^z)_t + u^r(u^z)_r + u^z(u^z)_z = -p_z + \nu\left[\frac{1}{r}(r(u^z)_r)_r + (u^z)_{zz}\right]. \tag{2.15c}$$

2.2.2 Plane flows

The general (non-classical) plane flow formulation is obtained by assuming planar symmetry, i.e. $z-$independence of all physical parameters, while keeping all velocity components generally nonzero. The equations describing general Navier-Stokes plane flows follow from the formulae (2.13) by choosing the parameters

$$a = 0, \quad b = 1, \quad B = r, \quad \xi = \varphi, \quad u^\xi \equiv u^\varphi, \quad u^\eta \equiv u^z \tag{2.16}$$

in terms of cylindrical coordinates (r, φ, z).

The classical (two-component) plane flow equations additionally assume no flow in the invariant direction, i.e., $u^z = 0$. In this setting, the equation for the z-projection of the momentum vanishes. It is more customary to present the resulting equations in Cartesian coordinates, where they take the form:

$$(u^x)_x + (u^y)_y = 0, \tag{2.17a}$$

$$(u^x)_t + u^x(u^x)_x + u^y(u^x)_y = -p_x + \nu\left[(u^x)_{xx} + (u^x)_{yy}\right], \tag{2.17b}$$

$$(u^y)_t + u^x(u^y)_x + u^y(u^y)_y = -p_y + \nu\left[(u^y)_{xx} + (u^y)_{yy}\right]. \tag{2.17c}$$

2.3 Equations in stream function formulation

In helical coordinates, the continuity equation (2.13a) is a two-dimensional divergence expression:

$$(ru^r)_r + \left(\frac{r}{B}u^\xi\right)_\xi = 0. \tag{2.18}$$

Hence, one can introduce a potential (*stream function*) $\Psi = \Psi(t, r, \xi)$ such that

$$u^r = -\frac{1}{r}\Psi_\xi, \quad u^\xi = \frac{B}{r}\Psi_r. \tag{2.19}$$

Substituting (2.19) into the helical Navier-Stokes equations (2.13) in order to eliminate the dependent variables u^r and u^ξ one obtains a potential system given by:

$$\begin{aligned}
&-\frac{1}{r}\Psi_{\xi t} + \frac{1}{r}\Psi_\xi\left(\frac{1}{r}\Psi_\xi\right)_r - \frac{1}{r^2}\Psi_r\Psi_{\xi\xi} - \frac{B^2}{r}\left(\frac{bB}{r^2}\Psi_r + au^\eta\right)^2 + p_r \\
&\qquad - \nu\left[-\left(\frac{1}{r}\Psi_\xi\right)_{rr} - \frac{1}{r}\left(\frac{1}{r}\Psi_\xi\right)_r - \frac{1}{rB^2}\Psi_{\xi\xi\xi}\right. \\
&\qquad\qquad \left. -\frac{1}{r^2}\left(-\frac{1}{r}\Psi_\xi + \frac{2b^2B^2}{r^2}\Psi_{\xi r} + 2abB\,(u^\eta)_\xi\right)\right] = 0,
\end{aligned} \tag{2.20a}$$

$$
\begin{aligned}
&(u^\eta)_t - \frac{1}{r}\Psi_\xi (u^\eta)_r + \frac{1}{r}\Psi_r (u^\eta)_\xi - \frac{a^2B^2}{r^2}u^\eta\Psi_\xi \\
&\quad - \nu B\left[\frac{B''}{B^2}u^\eta - \left(\frac{2abB}{r^3}\Psi_r + \left(\frac{b^2}{r^2}-a^2\right)\frac{u^\eta}{r}\right)B' + \frac{2B'}{B^2}(u^\eta)_r\right. \\
&\quad - \frac{2abB}{r^2}\left(\frac{B}{r}\Psi_r\right)_r - \left(\frac{b^2}{r^2}-a^2\right)\frac{B}{r}(u^\eta)_r + \frac{1}{B}(u^\eta)_{rr} + \frac{1}{B^3}(u^\eta)_{\xi\xi} \\
&\quad \left. + \frac{B}{r^2}\left(\frac{b^2}{r^2}-a^2\right)u^\eta - \frac{2ab}{r^3}\Psi_{\xi\xi}\right] = 0,
\end{aligned} \tag{2.20b}
$$

$$
\begin{aligned}
&\frac{B}{r}\Psi_{rt} - \frac{1}{r}\Psi_\xi\left(\frac{B}{r}\Psi_r\right)_r + \frac{B}{r^2}\Psi_r\Psi_{\xi r} - \frac{2abB^2}{r^3}u^\eta\Psi_\xi - \frac{b^2B^3}{r^5}\Psi_r\Psi_\xi + \frac{1}{B}p_\xi \\
&\quad - \nu B\left[\frac{B''}{rB}\Psi_r + \left(\frac{2abu^\eta}{r^2} - \left(\frac{b^2}{r^2}-a^2\right)\frac{B}{r^2}\Psi_r\right)B' + \frac{2B'}{B^2}\left(\frac{B}{r}\Psi_r\right)_r\right. \\
&\quad + \frac{2abB}{r^2}(u^\eta)_r - \left(\frac{b^2}{r^2}-a^2\right)\frac{B}{r}\left(\frac{B}{r}\Psi_r\right)_r + \frac{1}{B}\left(\frac{B}{r}\Psi_r\right)_{rr} \\
&\quad \left. + \frac{1}{rB^2}\Psi_{\xi\xi r} - \frac{2abB}{r^3}u^\eta - \frac{2b^2}{r^4}\Psi_{\xi\xi}\right] = 0.
\end{aligned} \tag{2.20c}
$$

A further reduction is obtained by elimination pressure p via cross-differentiation of the equations (2.20a) and (2.20c). The reduced system contains only two partial differential equations (orders 3 and 1, respectively):

$$
\begin{aligned}
&\left(\frac{B^2}{r}\Psi_{rr} + \frac{1}{r}\Psi_{\xi\xi} + \left(\frac{B^2}{r}\right)_r\Psi_r\right)_t + \frac{B^2}{r^2}\left(\Psi_{\xi rr}\Psi_r - \Psi_{rrr}\Psi_\xi\right) + \frac{1}{r^2}\left(\Psi_{\xi\xi\xi}\Psi_r - \Psi_{r\xi\xi}\Psi_\xi\right) \\
&\quad + \left(\frac{3a^2B^4}{r^3} - \frac{b^2B^4}{r^5}\right)\Psi_{rr}\Psi_\xi + \frac{2}{r^3}\Psi_{\xi\xi}\Psi_\xi - \left(\frac{a^2B^4}{r^3} - \frac{b^2B^4}{r^5}\right)\Psi_{\xi r}\Psi_r \\
&\quad + \left(\frac{a^2B^4}{r^3} - \frac{b^2B^4}{r^5}\right)_r\Psi_\xi\Psi_r + \frac{2abB^3}{r^3}\left(\Psi_r (u^\eta)_\xi - \Psi_\xi (u^\eta)_r\right) \\
&\quad - \left(\frac{2abB^3}{r^3}\right)_r\Psi_\xi u^\eta + \frac{2a^2B^2}{r}(u^\eta)_\xi u^\eta = \\
&\nu\left[\frac{B^2}{r}\Psi_{rrrr} + \frac{1}{B^2r}\Psi_{\xi\xi\xi\xi} + \frac{2}{r}\Psi_{rr\xi\xi} - \frac{4a^2B^4 - 2B^2}{r^2}\Psi_{rrr} - \frac{2}{r^2}\Psi_{r\xi\xi}\right. \\
&\quad + \left(\frac{B}{r}\left(\frac{B}{r}\right)_r + \left(\frac{B^2}{r^2}\right)_r + 2\left(B\left(\frac{B}{r}\right)_r\right)_r + B\left(\frac{B}{r}\right)_{rr} + \frac{a^4B^6 - B^2}{r^3}\right)\Psi_{rr} \\
&\quad - \left(\frac{2b^2B^2}{r^4}\right)_r\Psi_{\xi\xi} + \left(\left(\frac{B}{r}\left(\frac{B}{r}\right)_r\right)_r + \left(B\left(\frac{B}{r}\right)_{rr}\right)_r + \left(\frac{a^4B^6 - B^2}{r^3}\right)_r\right)\Psi_r \\
&\quad + \frac{2abB^3}{r^2}(u^\eta)_{rr} + \frac{2abB}{r^2}(u^\eta)_{\xi\xi} + \left(\left(\frac{2abB^3}{r^2}\right)_r + \frac{2abB^2}{r}\left(\frac{B}{r}\right)_r\right)(u^\eta)_r \\
&\quad \left. + \left(\frac{2abB^2}{r}\left(\frac{B}{r}\right)_r\right)u^\eta\right]
\end{aligned} \tag{2.21a}
$$

$$
\begin{aligned}
(u^\eta)_t - \frac{1}{r}\Psi_\xi (u^\eta)_r + \frac{1}{r}\Psi_r (u^\eta)_\xi - \frac{a^2B^2}{r^2}u^\eta\Psi_\xi \\
- \nu B\left[\frac{B''}{B^2}u^\eta - \left(\frac{2abB}{r^3}\Psi_r + \left(\frac{b^2}{r^2} - a^2\right)\frac{u^\eta}{r}\right)B' + \frac{2B'}{B^2}(u^\eta)_r\right. \\
- \frac{2abB}{r^2}\left(\frac{B}{r}\Psi_r\right)_r - \left(\frac{b^2}{r^2} - a^2\right)\frac{B}{r}(u^\eta)_r + \frac{1}{B}(u^\eta)_{rr} + \frac{1}{B^3}(u^\eta)_{\xi\xi} \\
\left. + \frac{B}{r^2}\left(\frac{b^2}{r^2} - a^2\right)u^\eta - \frac{2ab}{r^3}\Psi_{\xi\xi}\right] = 0. \quad (2.21b)
\end{aligned}
$$

2.3.1 The JFKO equation

The well-known JFKO (Johnson-Frieman-Kulsrud-Oberman) equation (Johnson et al. 1958) readily follows from the equations (2.20) in the case of time-independent inviscid flows:

$$
\frac{1}{r}\Psi_\xi\left(\frac{1}{r}\Psi_\xi\right)_r - \frac{1}{r^2}\Psi_r\Psi_{\xi\xi} - \frac{B^2}{r}\left(\frac{bB(r)}{r^2}\Psi_r + au^\eta\right)^2 + p_r = 0, \quad (2.22a)
$$

$$
-\frac{1}{r}\Psi_\xi (u^\eta)_r + \frac{1}{r}\Psi_r (u^\eta)_\xi - \frac{a^2B^2}{r^2}u^\eta\Psi_\xi = 0, \quad (2.22b)
$$

$$
-\frac{1}{r}\Psi_\xi\left(\frac{B}{r}\Psi_r\right)_r + \frac{B}{r^2}\Psi_r\Psi_{r\xi} - \frac{2abB^2}{r^3}u^\eta\Psi_\xi - \frac{b^2B^3}{r^5}\Psi_r\Psi_\xi + \frac{1}{B}p_\xi = 0. \quad (2.22c)
$$

Equation (2.22b) can be written in a more compact form as:

$$
\Psi_\xi\left(\frac{r}{B}u^\eta\right)_r - \Psi_r\left(\frac{r}{B}u^\eta\right)_\xi = 0
$$

or

$$
\left\{\Psi, \frac{r}{B}u^\eta\right\}_{\{r,\xi\}} = 0.
$$

The last identity is obtained using the Poisson bracket:

$$
\{F(x,y), G(x,y)\}_{\{x,y\}} := F(x,y)_x G(x,y)_y - F(x,y)_y G(x,y)_x.
$$

A zero Poisson bracket implies the functional dependence of the two functions within the bracket:

$$
\left\{\Psi, \frac{r}{B}u^\eta\right\}_{\{r,\xi\}} = 0 \quad \Rightarrow \quad u^\eta = \frac{B}{r}I(\Psi), \quad (2.23)
$$

where $I(\Psi)$ is an arbitrary function. Moreover, from equation (2.22a) and (2.22c) it follows that

$$
\left\{\Psi, p + \frac{1}{2}|\boldsymbol{u}|^2\right\}_{\{r,\xi\}} = 0,
$$

and subsequently,

$$
p + \frac{1}{2}|\boldsymbol{u}|^2 = -P(\Psi), \quad (2.24)
$$

where $P(\Psi)$ is a second arbitrary function. Finally, using (2.23) and (2.24) in (2.22) the helically symmetric time-independent Euler equations are equivalent to the resulting partial differential equation for the unknown function $\Psi(r, \xi)$, the JFKO equation:

$$B^2\Psi_{rr} - \frac{B^4}{r^3}\left(a^2r^2 - b^2\right)\Psi_r + \Psi_{\xi\xi} + B^2 I(\Psi)I'(\Psi) - 2ab\frac{B^4}{r^2}I(\psi) = -r^2P'(\Psi). \quad (2.25)$$

The time independent velocity vector is obtained from solutions of (2.25) through

$$\boldsymbol{u} = -\frac{1}{r}\Psi_\xi \boldsymbol{e}_r + \frac{B}{r}I(\Psi)\boldsymbol{e}_\eta + \frac{B}{r}\Psi_r \boldsymbol{e}_\xi. \quad (2.26)$$

The Bragg-Hawthorne or Grad-Rubin-Shafranov equation for axially symmetric flows (Bragg & Hawthorne (1950), Grad & Rubin (1958), Shafranov (1958)) is obtained from the JFKO equation (2.25) using the parameter set in (2.14):

$$\Psi_{rr} - \frac{1}{r}\Psi_r + \Psi_{\xi\xi} + I(\Psi)I'(\Psi) = -r^2P'(\Psi). \quad (2.27)$$

2.4 Equations in vorticity formulation

The vorticity formulation of the Navier-Stokes equations (2.8) consists of the continuity equation, the definition of vorticity, and the vorticity dynamics equation obtained by taking the curl of the momentum equation (2.8b). It has the form:

$$\nabla \cdot \boldsymbol{u} = 0, \quad (2.28a)$$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad (2.28b)$$

$$\boldsymbol{\omega}_t + \nabla \times \left(\boldsymbol{\omega} \times \boldsymbol{u}\right) - \nu\nabla^2\boldsymbol{\omega} = 0. \quad (2.28c)$$

In the helical basis, the vorticity vector $\boldsymbol{\omega}$ is given by:

$$\boldsymbol{\omega} = \omega^r \boldsymbol{e}_r + \omega^\eta \boldsymbol{e}_{\perp\eta} + \omega^\xi \boldsymbol{e}_\xi. \quad (2.29)$$

Under the assumption of helical invariance, the respective components of $\boldsymbol{\omega}$ are given by:

$$\begin{aligned} \boldsymbol{\omega} &= \nabla \times \boldsymbol{u} \\ &= \left[\frac{1}{r}\left(u^z\right)_\varphi - \left(u^\varphi\right)_z\right]\boldsymbol{e}_r + \left[\left(u^r\right)_z - \left(u^z\right)_r\right]\boldsymbol{e}_\varphi + \frac{1}{r}\left[\left(ru^\varphi\right)_r - \left(u^r\right)_\varphi\right]\boldsymbol{e}_z \\ &= \left[\frac{b}{r}\left(u^z\right)_\xi - a\left(u^\varphi\right)_\xi\right]\boldsymbol{e}_r + \left[a\left(u^r\right)_\xi - \left(u^z\right)_r\right]\boldsymbol{e}_\varphi + \frac{1}{r}\left[\left(ru^\varphi\right)_r - b\left(u^r\right)_\xi\right]\boldsymbol{e}_z. \end{aligned} \quad (2.30)$$

The step from the second to the third line of the equation (2.30) contains the application of the transformation rule for the derivatives (2.7). Further, one has to replace the cylindrical velocities and the unit vectors by the helical ones using the relations (2.12), to obtain the final result:

$$\begin{aligned}\boldsymbol{\omega} = &\left[-\frac{1}{B}\left(u^\eta\right)_\xi\right]\boldsymbol{e}_r \\ &+\left[\frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(ru^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi\right]\boldsymbol{e}_\eta + \left[(u^\eta)_r + \frac{a^2B^2}{r}u^\eta\right]\boldsymbol{e}_\xi \end{aligned} \tag{2.31}$$

with

$$\omega^r = -\frac{1}{B}(u^\eta)_\xi, \tag{2.32a}$$

$$\omega^\eta = \frac{1}{B}(u^r)_\xi - \frac{1}{r}\left(ru^\xi\right)_r - \frac{2abB^2}{r^2}u^\eta + \frac{a^2B^2}{r}u^\xi, \tag{2.32b}$$

$$\omega^\xi = (u^\eta)_r + \frac{a^2B^2}{r}u^\eta. \tag{2.32c}$$

The helically invariant reduction of the three projections of the vorticity equation (2.28c) yields the three partial differential equations:

$$\begin{aligned}(\omega^r)_t + u_r(\omega^r)_r + \frac{1}{B}u^\xi(\omega^r)_\xi = \omega^r(u^r)_r + \frac{1}{B}\omega^\xi(u^r)_\xi \\ + \nu\left[\frac{1}{r}(r(\omega^r)_r)_r + \frac{1}{B^2}(\omega^r)_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left(a(\omega^\eta)_\xi + \frac{b}{r}(\omega^\xi)_\xi\right)\right],\end{aligned} \tag{2.33a}$$

$$\begin{aligned}&(\omega^\eta)_t + u^r(\omega^\eta)_r + \frac{1}{B}u^\xi(\omega^\eta)_\xi \\ &\quad - \frac{a^2B^2}{r}(u^r\omega^\eta - u^\eta\omega^r) + \frac{2abB^2}{r^2}(u^\xi\omega^r - u^r\omega^\xi) = \omega^r(u^\eta)_r + \frac{1}{B}\omega^\xi(u^\eta)_\xi \\ &+ \nu\left[\frac{1}{r}(r(\omega^\eta)_r)_r + \frac{1}{B^2}(\omega^\eta)_{\xi\xi} + \frac{a^2B^2(a^2B^2-2)}{r^2}\omega^\eta + \frac{2abB}{r^2}\left((\omega^r)_\xi - \left(B\omega^\xi\right)_r\right)\right],\end{aligned} \tag{2.33b}$$

$$\begin{aligned}&(\omega^\xi)_t + u^r(\omega^\xi)_r + \frac{1}{B}u^\xi(\omega^\xi)_\xi \\ &\quad + \frac{1-a^2B^2}{r}(u^\xi\omega^r - u^r\omega^\xi) = \omega^r(u^\xi)_r + \frac{1}{B}\omega^\xi(u^\xi)_\xi \\ &+ \nu\left[\frac{1}{r}(r(\omega^\xi)_r)_r + \frac{1}{B^2}(\omega^\xi)_{\xi\xi} + \frac{a^4B^4-1}{r^2}\omega^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}(\omega^r)_\xi + \left(\frac{aB}{r}\omega^\eta\right)_r\right)\right].\end{aligned} \tag{2.33c}$$

The first two terms on the right-hand side of each equation in (2.33a)-(2.33c) correspond to vortex stretching. The presence of these terms is not obvious, even though one has three velocity components, one has to recall the fact, that the system of equations depend only on two spatial coordinates instead of three.

At this point it is worth noting that in three dimensions, the vorticity vector $\boldsymbol{\omega}$ is a locally conserved quantity, since all three components of the vorticity equation (2.28c) are indeed divergence expressions, with components of $\boldsymbol{\omega}$ being the conserved densities. However, it is not a material conservation law, due to the vortex stretching. To the best of the authors knowledge, a vorticity-related material conservation law was only known for plane flows with a transverse velocity component of zero and for axisymmetric flows. In chapter 6, new material conservation laws will be derived for two-component inviscid helically symmetric flows that essentially involve vorticity.

In the presentation of local conservation laws involving vorticity, in order to simplify expressions, sometimes the cylindrical vorticity components will be used. The relations between helical and cylindrical components are (similarly to the velocities in (2.11)):

$$\omega^\eta = B\left(a\omega^\varphi - \frac{b}{r}\omega^z\right), \qquad \omega^\xi = B\left(\frac{b}{r}\omega^\varphi + a\omega^z\right). \tag{2.34}$$

The reverse relations are given by (similarly to (2.12)):

$$\omega^\varphi = B\left(a\omega^\eta + \frac{b}{r}\omega^\xi\right), \qquad \omega^z = B\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right). \tag{2.35}$$

3 Direct construction of conservation laws

One may think of Emmy Noether and her famous work (Noether 1918) when conservation laws are mentioned. In her theorem 1918 Emmy Noether stated that any differentiable symmetry of the action of a physical system has a corresponding conservation law. The action of a physical system is the integral over time of a Lagrangian function, from which the behaviour of the system can be determined by the principle of least action. The need of an Lagrangian, however, makes this theorem only applicable to a restricted set of problems, in particular it can not be applied to dissipative systems. For this reason a different method, which will be described in this chapter, will be used for finding conservation laws.

For any given system of partial differential equations, one can seek its divergence-type conservation laws (1.1), where the conserved density Θ and spatial fluxes Φ^i, $i = 1, 2, 3$ may depend on independent and dependent variables of the given equations, on partial derivatives of dependent variables, and perhaps also on nonlocal (integral) quantities. A brief overview of the algorithm behind the method of direct construction of local conservation laws will be provided here. For further details, see, e.g., Anco et al. (2010).

The direct method consists of essentially two key ideas. Consider a system

$$R^\sigma\left[u\right] = R^\sigma\left(z, u, \partial u, ..., \partial^k u\right) = 0, \quad \sigma = 1, \dots, N, \tag{3.1}$$

of N partial differential equations, with n independent variables $z = (z^1, \dots, z^n)$ (one of which can be time) and m dependent variables $u = (u^1, \dots, u^m)$. The direct construction method seeks conservation laws in the form

$$\frac{\partial \Gamma^i}{\partial z^i} = 0, \tag{3.2}$$

which is equivalent to (1.1). Let

$$\mathcal{E}_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \dots i_s}} + \cdots \tag{3.3}$$

denote the Euler differential operator with respect to each dependent variable u^j, where D_i is a total derivative operator with respect to z^i defined as:

$$D_i = \frac{\partial}{\partial z^i} + u^j_i \frac{\partial}{\partial u^j} + u^j_{ii_1} \frac{\partial}{\partial u^j_{i_1}} + u^j_{ii_1 i_2} \frac{\partial}{\partial u^j_{i_1 i_2}} + \cdots , \tag{3.4}$$

and $u^j_{i_1...i_s} \equiv \partial^s u^j/\partial z^{i_1} \dots \partial z^{i_s}$ is a partial derivative of order s.
A very useful property of the Euler operator is the following: an expression F depending on z, u, and derivatives of u, is annihilated by an Euler operator with respect to each u^j,

$$\mathcal{E}_{u^j}(F) \equiv 0, \quad j = 1, \dots, m, \tag{3.5}$$

if, and only if, F is in divergence form such as the left hand side of equation (3.2) (Anco et al. 2010). This is in fact the first essential idea of the construction scheme. Note that in (3.5), functions u_j are arbitrary, and are not restricted to be solutions of the given equations (3.1).

The second main idea relies on the fact that the direct construction method searches for conservation laws as linear combinations of the given equations R^σ from (3.1) with unknown multipliers Λ_σ:

$$\Lambda_\sigma R^\sigma \equiv \frac{\partial \Gamma^i}{\partial z^i} = 0. \tag{3.6}$$

The multipliers may be chosen to depend on independent and dependent variables and partial derivatives of the dependent variables, up to some prescribed order. From (3.5) it follows that the multipliers must satisfy the multiplier determining equations

$$\mathcal{E}_{u^j}(\Lambda_\sigma R^\sigma) = 0, \quad j = 1, ..., m. \tag{3.7}$$

After the linear determining equations (3.7) are solved and multipliers Λ_σ are found, one proceeds with the finding of conservation law density and fluxes Γ^i, using (3.6). For further details, see, e.g., Anco et al. (2010).

It should be noted that if the considered PDE system possesses a Lagrangian, the multipliers are the symmetries of this system of equations (Noether's theorem).

It is important to note that the majority of PDE systems arising in applications, such as Euler equations, can be written in a solved form with respect to some leading derivatives. It has been proven that for such systems, all of their local conservation laws can be found in the form (3.6). Moreover, for Cauchy-Kovalevskaya PDE systems (systems solved with respect to highest derivatives of all dependent variables with respect to some independent variable), there is a one-to-one correspondence between sets of conservation law multipliers and conservation laws themselves (Anco et al. 2010). It is evident that the helically invariant Euler equations, i.e. (2.13) with $\nu = 0$, can be written in a Cauchy-Kovalevskaya form with respect to r, whereas the helically invariant Navier-stokes equations (2.13) do not have a Cauchy-Kovalevskaya form.

In the computations that employ the direct construction method, one naturally avoids trivial conservation laws, which can arise as differential identities such as $\nabla \cdot (\nabla \times (\cdot)) \equiv 0$, or alternatively as "$0 = 0$" conservation laws, whose density and all fluxes vanish identically on solutions of the given system.

For complicated PDE systems, such as equations of fluid dynamics considered in the current dissertation, multiplier determining equations (3.7) lead to a system containing thousands of overdetermined linear PDEs on $\{\Lambda_\sigma\}$. In order to perform these computations, a symbolic software package GeM for Maple (Cheviakov 2007) and the powerful Maple rifsimp routine for differential polynomial system reduction were intensively used. (Please note, that after a specific conservation law is obtained, its correctness can be verified directly by hand, without any specialized software.)

As noted above, the direct construction method is used to discover families of conservation laws of the helical reductions of Euler and Navier-Stokes equations in primitive variables as well as in alternative formulations.

In this work one seeks conservation laws in the canonical form (1.1), which in the helically symmetric setting becomes:

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{\Phi} \equiv \frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Phi^r\right) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0. \tag{3.8}$$

The direct construction method yields divergence expressions (3.2), which can be converted to the canonical form (3.8) by the transformation

$$\frac{\partial \Gamma^1}{\partial t} + \frac{\partial \Gamma^2}{\partial r} + \frac{\partial \Gamma^3}{\partial \xi} = r\left[\frac{\partial}{\partial t}\left(\frac{\Gamma^1}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\Gamma^2}{r}\right) + \frac{1}{B}\frac{\partial}{\partial \xi}\left(\frac{B}{r}\Gamma^3\right)\right] = 0, \tag{3.9}$$

that is,

$$\Theta \equiv \frac{\Gamma^1}{r}, \quad \Phi^r \equiv \frac{\Gamma^2}{r}, \quad \Phi^\xi \equiv \frac{B}{r}\Gamma^3.$$

In the following chapters, conservation laws (3.8), which can be identified with material conservation laws (1.3), will be pointed out.

4 Conservation laws of the helically invariant Euler system

The direct construction method will be applied to seek local conservation laws of the helically invariant Euler equations in primitive variables and in vorticity formulation. In primitive variables, the conservation law multipliers Λ_σ in (3.6) and (3.7) were chosen to depend on all independent and dependent variables and first partial derivatives of the dependent variables of the system:

$$\Lambda_\sigma = \Lambda_\sigma \left(t, r, \xi, u^r, u^\eta, u^\xi, p, (u^r)_\xi, (u^\eta)_r, (u^\eta)_\xi, (u^\xi)_r, (u^\xi)_\xi, p_t, p_r, p_\xi\right).$$

In the vorticity formulation, multipliers were restricted to depend on all independent and dependent variables of the vorticity system:

$$\Lambda_\sigma = \Lambda_\sigma \left(t, r, \xi, u^r, u^\eta, u^\xi, p, \omega^r, \omega^\eta, \omega^\xi\right).$$

More complicated forms of multipliers resulted in intractable multiplier determining equations even with the aid of the computer algebra software.

In the current, as well as in the following chapters, the density Θ and the fluxes Φ^r, Φ^ξ of the conservation laws are listed in the form (3.8). For simplicity and compactness of the presentation, both the helical and the cylindrical notation for velocity and vorticity components will be used, as per (2.11), (2.12), (2.34) and (2.35).

The two obvious conservation laws $\nabla \cdot (G(t)\boldsymbol{u}) = 0$ and $\nabla \cdot (G(t)\boldsymbol{\omega}) = 0$ that hold for an arbitrary function $G(t)$ and reflect the obvious scaling properties of the continuity equations $\nabla \cdot \boldsymbol{u} = 0$, $\nabla \cdot \boldsymbol{\omega} = 0$ will not be explicitly listed.

Results for the stream function formulation are not presented below, since no additional conservation laws have been found that arise from it.

4.1 Primitive variables

The helically invariant Euler system in primitive variables is given by formulae (2.13) with $\nu = 0$. The conservation laws obtained from this system are denoted by the prefix "EP". Conservation laws arising from the Euler vorticity system (equations (2.33) with $\nu = 0$) are denoted by the prefix "EV".

EP1. Conservation of kinetic energy. The conservation law is given by:

$$\Theta = K, \quad \Phi^r = u^r(K + p), \quad \Phi^\xi = u^\xi(K + p), \tag{4.1}$$

where K is the kinetic energy density given by:

$$K = \frac{1}{2}|\boldsymbol{u}|^2 = \frac{1}{2}\left((u^r)^2 + (u^\eta)^2 + \left(u^\xi\right)^2\right).$$

EP2. Conservation of the z-projection of momentum. It is well known that for Euler equations, every projection of momentum in Cartesian coordinates is conserved, however, this is generally not the case for momentum projections in curvilinear coordinates. In helical coordinates with imposed helical invariance, the z-projection of momentum is the only locally conserved quantity. The density and the fluxes of the corresponding conservation law are given by:

$$\Theta = B\left(-\frac{b}{r}u^\eta + au^\xi\right) = u^z, \quad \Phi^r = u^r u^z, \quad \Phi^\xi = u^\xi u^z + aBp. \tag{4.2}$$

The conservation law (4.2) yields a material conservation law

$$\frac{du^z}{dt} = 0$$

when $\partial p/\partial \xi = 0$, i.e., when $p = p(r,t)$.

EP3. Conservation of the z-projection of the angular momentum. In a similar fashion, the z-projection of the angular momentum is conserved:

$$\Theta = rB\left(au^\eta + \frac{b}{r}u^\xi\right) = ru^\varphi, \quad \Phi^r = ru^r u^\varphi, \quad \Phi^\xi = ru^\xi u^\varphi + bBp. \tag{4.3}$$

It also yields a material conservation law:

$$\frac{d(ru^\varphi)}{dt} = 0$$

for $\partial p/\partial \xi = 0$.

EP4. Conservation of the generalized momenta/angular momenta. In helical coordinates, neither momentum nor the angular momentum in the directions η or ξ are conserved; however the helically invariant Euler equations possess an infinite family of conservation laws given by:

$$\Theta = F\left(\frac{r}{B}u^\eta\right), \quad \Phi^r = u^r F\left(\frac{r}{B}u^\eta\right), \quad \Phi^\xi = u^\xi F\left(\frac{r}{B}u^\eta\right), \tag{4.4}$$

where $F(\cdot)$ is an arbitrary function.

In order to give a physical interpretation to the conservation laws (4.4), one uses (2.11) to get

$$\zeta = \frac{r}{B}u^\eta = aru^\varphi - bu^z.$$

The quantity ζ may be interpreted as a "blend" of momentum and angular momentum density in the $\eta-$direction. Indeed, in the limiting case of planar symmetry when $a = 0$, one has $\zeta \sim u^z$, which is proportional to the linear momentum density in the z-direction. In the rotationally symmetric case when $b = 0$, one gets $\zeta \sim ru^\varphi$, which is proportional to the angular momentum density in z-direction. (The dimensional consistency is provided through the physical dimensions of constants a, b.) Consequently, in a special case $F(\zeta) = \zeta$, the "momentum blend" ζ is the conserved quantity; in the case of the general $F(\zeta)$, one has an infinite set of "generalized momenta/angular momenta" conservation laws.

It should be noted that all conservation laws (4.4) are material conservation laws:

$$\frac{d}{dt} F\left(\frac{r}{B}u^\eta\right) = 0.$$

The existence of the present family of material conservation laws for inviscid flows, involving a free function, is related to the fact that the momentum equation in the direction of invariance decouples from the system, and the pressure gradient in the respective directions vanishes. As a result the equation becomes a first-order PDE linear in ζ. Such equations admit a relabeling symmetry, which here takes the form

$$\frac{r}{B}u^\eta \to F\left(\frac{r}{B}u^\eta\right) \tag{4.5}$$

which follows from multiplying the mentioned linear equation by $F'\left(\frac{r}{B}u^\eta\right)$ to obtain the conservation law (4.4). A similar property is well known for the vorticity conservation of planar two-component flows (section 6.2) and for the vorticity conservation of axisymmetric flows (section 6.3).

4.2 The vorticity formulation

In this section one considers the conservation laws derived from continuity and momentum equations (2.13) extended by the vorticity transport equations (2.33) with $\nu = 0$ and the definition of the vorticity given by (2.32). From this extended system additional families of conservation laws are to be expected. Similar to the momentum equation, only a part of the vorticity conservation itself will be retained but further we observe generalized helicity which is an entanglement of velocity and vorticity and various new vorticity related conservation laws are derived.

Further, it is trivial that all previously derived conservation laws derived in subsection 4.1 carry over to the presently extended system.

EV1. Conservation of helicity. Most naturally one expects the conservation of helicity

$$h = \boldsymbol{u} \cdot \boldsymbol{\omega} = u^r \omega^r + u^\eta \omega^\eta + u^\xi \omega^\xi$$

which also in three dimensions follows from the Euler system extended by the vorticity formulation. The conservation law is given by:

$$
\begin{aligned}
\Theta &= h,\\
\Phi^r &= \omega^r\left(E-(u^\eta)^2-\left(u^\xi\right)^2\right)+u^r\left(h-u^r\omega^r\right),\\
\Phi^\xi &= \omega^\xi\left(E-(u^r)^2-(u^\eta)^2\right)+u^\xi\left(h-u^\xi\omega^\xi\right),
\end{aligned}
\tag{4.6}
$$

where

$$
E=\frac{1}{2}|\boldsymbol{u}|^2+p=\frac{1}{2}\left((u^r)^2+(u^\eta)^2+\left(u^\xi\right)^2\right)+p \tag{4.7}
$$

is the total energy density.

In vector notation, the helicity conservation law (4.6) can be written as

$$
\frac{\partial}{\partial t}h+\nabla\cdot\left(\boldsymbol{u}\times\nabla E+(\boldsymbol{\omega}\times\boldsymbol{u})\times\boldsymbol{u}\right)=0. \tag{4.8}
$$

EV2. An infinite family of generalized helicity conservation laws. Interestingly, for helically invariant inviscid flows, it was found that the conservation of helicity (4.6) can be vastly generalized. The following family of conservation laws holds, involving an arbitrary function $H=H\left(\frac{r}{B}u^\eta\right)$:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(h\,H\left(\frac{r}{B}u^\eta\right)\right)\\
&\quad+\nabla\cdot\left[H\left(\frac{r}{B}u^\eta\right)\left[\boldsymbol{u}\times\nabla E+(\boldsymbol{\omega}\times\boldsymbol{u})\times\boldsymbol{u}\right]\right.\\
&\qquad\left.+\,Eu^\eta\boldsymbol{e}_{\perp\eta}\times\nabla H\left(\frac{r}{B}u^\eta\right)\right]=0.
\end{aligned}
\tag{4.9}
$$

For $H=1$, (4.9) reduces to the conservation of helicity (4.8).

It is evident that the arbitrary functions in the formula (4.9) and in the generalized momentum conservation laws (4.4) have the same argument $\frac{r}{B}u^\eta$. An important difference, however, is that unlike the generalized momentum conservation laws (4.4), the generalized helicity conservation laws (4.9) essentially involve all three velocity and vorticity components.

Unlike the generalized momentum case (4.4), the free function H in formula (4.9) does not arise due to a relabeling symmetry (4.5), even though the expression (4.9) is linear in H. Moreover, the generalized helicity conservation laws (4.9) do not correspond to a material conservation law.

It follows that from a physical point of view, the present family of conservation laws is clearly distinguished from all other known types of conservation laws known in fluid dynamics where free functions depend on the dependent variables.

Material conservation laws such as the generalized momentum laws (4.4), which involve arbitrary functions due to a relabeling symmetry, do not really describe a different physical quantity for different choices of the form of the arbitrary functions; instead, they in some sense are "different sides of the same dice", describing conservation of the same infinitesimal quantity related to a fluid parcel. For the generalized helicity conservation laws (4.9), the situation is intrinsically different, since every different choice of the free function H brings up a different physical flow quantity with its individual flow dynamics. The fact that helically symmetric inviscid flows admit an infinite number of independent conservation laws is fundamentally unique; their existence can be interpreted as a manifestation of the simplified flow geometry, in which they are only known to arise.

EV3. A family of vorticity conservation laws involving ω^φ. A family of conservation laws is given by:

$$\begin{aligned} \Theta &= \frac{Q(t)}{r}\omega^\varphi, \\ \Phi^r &= \frac{1}{r}\left(Q(t)[u^r\omega^\varphi - \omega^r u^\varphi] + Q'(t)u^z\right), \\ \Phi^\xi &= -\frac{aB}{r}\left(Q(t)\left[u^\eta\omega^\xi - u^\xi\omega^\eta\right] + Q'(t)u^r\right), \end{aligned} \tag{4.10}$$

where $Q(t)$ is an arbitrary function.

The following two conservation laws are specific to the helical geometry of the flow and do not correspond to material conservation laws. It turns out they hold both for the Euler equations and the Navier-Stokes equations, as will be seen in chapter 5.

EV4. Vorticity conservation law (i). The conservation law is given by:

$$\begin{aligned} \Theta &= -rB\left(a^3\omega^\eta - \frac{b^3}{r^3}\omega^\xi\right), \\ \Phi^r &= -2a^2u^ru^z - a^3Br\left(u^r\omega^\eta - u^\eta\omega^r\right) + \frac{Bb^3}{r^2}\left(u^r\omega^\xi - u^\xi\omega^r\right), \\ \Phi^\xi &= a^3B\left[(u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right] + \frac{2a^2bB}{r}u^\eta u^\xi. \end{aligned} \tag{4.11}$$

In both the rotationally symmetric setting $a = 1, b = 0$ and the plane symmetry setting $a = 0, b = 1$, the conserved quantity Θ is related to the polar vorticity component. In the plane case, it reduces to $\Theta = \omega^\varphi/r$ and becomes a part of the family (4.10); in the rotationally symmetric case, one has $\Theta = -r\omega^\varphi$. For problems where the flow velocity vanishes on the boundary of the flow domain Ω, the quantity $r\omega^\varphi$ corresponds to the conservation of linear momentum in the z-direction, since

$$\frac{1}{2}\iint_\Omega r\omega^\varphi dA = \iint_\Omega u^z dA. \tag{4.12}$$

In the general helically symmetric setting $a, b \neq 0$, the conservation law (4.11) is independent of all other listed conservation laws.

EV5. Vorticity conservation law (ii). An additional conservation law involving two vorticity components is given by:

$$
\begin{aligned}
\Theta =& -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + a^3r^4\left(-\frac{b}{r}\omega^\eta + a\omega^\xi\right)\right) = -\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi + \frac{a^3r^4}{B}\omega^z\right),\\
\Phi^r =& a^3rB\left(2u^r\left(au^\eta + \frac{b}{r}u^\xi\right) + b\left(u^r\omega^\eta - u^\eta\omega^r\right)\right)\\
&- \frac{a^4r^4 + a^2r^2b^2 + b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi - u^\xi\omega^r\right),\\
\Phi^\xi =& -a^3bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4rBu^\eta u^\xi.
\end{aligned}
\tag{4.13}
$$

To have some insight into the structure of the conserved density in (4.13), one again considers the limiting cases. In the rotationally symmetric case $a = 1, b = 0$, one has $\xi = z$, and the conserved quantity in (4.13) reduces to $\Theta = -r^2\omega^z$. For problems where the flow velocity vanishes on the boundary of the flow domain Ω, the quantity $r^2\omega^z$ corresponds to the conservation of the angular momentum in z-direction, similarly to (4.12). In the plane symmetry case $a = 0, b = 1$, one has $\xi = \varphi$, with the conserved density becoming $\Theta = -\omega^\varphi/r$, which is again a part of the family (4.10). In the general case of helical symmetry with $a, b \neq 0$, however, the conservation law (4.13) is independent of all other listed conservation laws.

EV6. Vorticity conservation law (iii). A family of purely spatial divergence expressions that hold for both Euler and Navier-Stokes helically invariant equations in vorticity formulation is given by:

$$
\nabla\cdot\mathbf{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi, \tag{4.14}
$$

for an arbitrary function $N = N(t, \xi)$. This is a generalization of the obvious divergence expression $\nabla\cdot(G(t)\boldsymbol{\omega}) = 0$ that holds only for helically invariant flows.

5 Conservation laws of the helically invariant Navier-Stokes system

In the present chapter, the conservation laws derived by applying the direct construction method to the helically symmetric Navier-Stokes equations in primitive variables and vorticity formulation, with conservation law multipliers Λ depending on independent variables t, r, ξ, the physical parameters and their derivatives are listed.

One may generally note that all conservation laws which are subsequently derived for the helically symmetric Navier-Stokes equations are a subset of those admitted by the helically symmetric Euler equations, in the sense that the density is identical, while the fluxes are extended with the additional viscous terms. Remarkably, the helical Navier-Stokes equations share with the helical Euler equations the infinite families of conservation laws involving arbitrary functions.

The conservation of helicity and helicity-related quantities given by (4.8) and (4.9) does not hold for the viscous case.

5.1 Primitive variables

NSP1. Conservation of the z-projection of momentum. The following conservation law

$$\Theta = u^z, \quad \Phi^r = u^r u^z - \nu (u^z)_r, \quad \Phi^\xi = u^\xi u^z + aBp - \frac{\nu}{B}(u^z)_\xi \tag{5.1}$$

generalizes the conservation of momentum for the helical Euler system, given by (4.2).

NSP2. Conservation of a generalized momentum. The following conservation law is a viscous extension of the conservation laws (4.4):

$$\begin{aligned}
\Theta &= \frac{r}{B}u^\eta, \\
\Phi^r &= \frac{r}{B}u^r u^\eta - \nu\left[-2aB\left(au^\eta + 2\frac{b}{r}u^\xi\right) + \left(\frac{r}{B}u^\eta\right)_r\right] \\
&= \frac{r}{B}u^r u^\eta - \nu\left[-2au^\varphi + \left(\frac{r}{B}u^\eta\right)_r\right], \\
\Phi^\xi &= \frac{r}{B}u^\eta u^\xi - \nu\frac{1}{B}\left[\frac{2abB^2}{r}u^r + \left(\frac{r}{B}u^\eta\right)_\xi\right],
\end{aligned} \tag{5.2}$$

where instead of an infinite "generalized momentum" family, only one conservation law holds. It corresponds to the extension of the "generalized momentum" conservation law (4.4) with $F\left(\frac{r}{B}u^\eta\right) = \frac{r}{B}u^\eta$ onto the viscous case.

5.2 The vorticity formulation

NSV1. An infinite family of vorticity conservation laws (i). The family of conservation laws (4.10) in the inviscid case is carried over to the viscous case, as follows:

$$
\begin{aligned}
\Theta =&\frac{Q(t)}{r}B\left(a\omega^\eta+\frac{b}{r}\omega^\xi\right)=\frac{Q(t)}{r}\omega^\varphi,\\
\Phi^r =&\frac{1}{r}\left\{Q(t)\left[u^rB\left(a\omega^\eta+\frac{b}{r}\omega^\xi\right)-\omega^rB\left(au^\eta+\frac{b}{r}u^\xi\right)\right]+Q'(t)B\left(-\frac{b}{r}u^\eta+au^\xi\right)\right.\\
&\left.-Q(t)\nu\left[\frac{aB}{r}\omega^\eta+\frac{b^2B}{r\left(a^2r^2+b^2\right)}\left(a\omega^\eta+\frac{b}{r}\omega^\xi\right)+B\left(a\omega^\eta_r+\frac{b}{r}\omega^\xi_r\right)\right]\right\},\\
\Phi^\xi =&-\frac{B}{r}\left\{aQ(t)\left[u^\eta\omega^\xi-u^\xi\omega^\eta\right]+aQ'(t)u^r+\frac{Q(t)}{r^3}\nu\left[\frac{r^3}{B}\left(a\omega^\eta_\xi+\frac{b}{r}\omega^\xi_\xi\right)+2br\omega^r\right]\right\},
\end{aligned}
\tag{5.3}
$$

where $Q(t)$ is an arbitrary function.

NSV2. Vorticity conservation law (ii). Likewise, the conservation law (4.11) for the inviscid case extends to its viscous form:

$$
\begin{aligned}
\Theta =&-rB\left(a^3\omega^\eta-\frac{b^3}{r^3}\omega^\xi\right),\\
\Phi^r =&-\frac{B}{r^2}\left(a^3r^3\left(u^r\omega^\eta-u^\eta\omega^r\right)-b^3\left(u^r\omega^\xi-u^\xi\omega^r\right)\right)-2a^2Bu^r\left(-\frac{b}{r}u^\eta+au^\xi\right)\\
&-\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(a\omega^\eta+\frac{b}{r}\omega^\xi\right)-r^3\left(a^3\omega^\eta_r-\frac{b^3}{r^3}\omega^\xi_r\right)+abB^2r\left(\frac{b^3}{r^3}\omega^\eta+a^3\omega^\xi\right)\right],\\
\Phi^\xi =&a^3B\left((u^r)^2+(u^\eta)^2-(u^\xi)^2+r\left(u^\eta\omega^\xi-u^\xi\omega^\eta\right)\right)+\frac{2a^2bB}{r}u^\eta u^\xi\\
&+\frac{2a^2bB}{r}\nu\left[\left(1-\frac{b^2}{a^2r^2}\right)\omega^r+\frac{r^2}{2a^2bB}\left(a^3\omega^\eta_\xi-\frac{b^3}{r^3}\omega^\xi_\xi\right)\right].
\end{aligned}
\tag{5.4}
$$

NSV3. Vorticity conservation law (iii). Similarly, the conservation law (4.13) holds for the Navier-Stokes formulation, with spatial fluxes modified as follows:

$$
\begin{aligned}
\Theta =&-\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi+a^3r^4\left(-\frac{b}{r}\omega^\eta+a\omega^\xi\right)\right)=-\frac{B}{r^2}\left(\frac{b^2r^2}{B^2}\omega^\xi+\frac{a^3r^4}{B}\omega^z\right),\\
\Phi^r =&a^3rB\left(2u^r\left(au^\eta+\frac{b}{r}u^\xi\right)+b\left(u^r\omega^\eta-u^\eta\omega^r\right)\right)-\frac{a^4r^4+a^2r^2b^2+b^4}{r\sqrt{a^2r^2+b^2}}\left(u^r\omega^\xi-u^\xi\omega^r\right)\\
&+\nu\left[4a^3B\left(au^\eta+\frac{b}{r}u^\xi\right)-a^3brB(\omega^\eta)_r+\frac{B}{r^3}\left(b^4-a^4r^4-\frac{a^6r^6}{a^2r^2+b^2}\right)\omega^\xi\right.\\
&\left.+\frac{B}{r^2}\left(a^4r^4+a^2r^2b^2+b^4\right)(\omega^\xi)_r+\frac{ab}{B}\left(2+\frac{a^4r^4}{(a^2r^2+b^2)^2}\right)\omega^\eta\right],
\end{aligned}
$$

$$\Phi^\xi = -\,a^3bB\left((u^r)^2 + (u^\eta)^2 - (u^\xi)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + 2a^4rBu^\eta u^\xi \\ +\,\nu\left[\frac{1}{r^2}\left(a^4r^4 + a^2r^2b^2 + b^4\right)(\omega^\xi)_\xi - a^3br(\omega^\eta)_\xi - \frac{4a^3bB}{r}u^r + \frac{2b^4B}{r^3}\omega^r\right]. \tag{5.5}$$

NSV4. Vorticity conservation law (iv). The family of spatial divergence expressions (4.14) corresponding to the generalization of vorticity continuity equation holds in the viscous case without change:

$$\nabla\cdot\mathbf{\Phi} = 0, \quad \Phi^r = N\omega^r - \frac{1}{B}N_\xi u^\eta, \quad \Phi^\xi = N\omega^\xi, \tag{5.6}$$

for an arbitrary function $N = N(t,\xi)$.

6 Extended sets of conservation laws for two-component flows

For both cases of a plane and an axisymmetric flow, it is well known that the vanishing of the velocity in the direction of the invariance leads to an extended set of conservation laws; moreover, certain conservation laws only exist for this special ansatz. The most well known example is that of a classical plane flow, which admits an infinite number of vorticity conservation laws (Bowman 2009).

In the present chapter one seeks to extend the classes of conservation laws admitted by helically invariant inviscid and viscous flow equations for the two-component flow, i.e.,

$$u^\eta = 0. \tag{6.1}$$

The helically invariant Navier-Stokes equations (2.13) consequently become

$$\frac{1}{r}u^r + (u^r)_r + \frac{1}{B}(u^\xi)_\xi = 0, \tag{6.2a}$$

$$\begin{aligned}(u^r)_t + u^r(u^r)_r + \frac{1}{B}u^\xi(u^r)_\xi - \frac{b^2B^2}{r^3}(u^\xi)^2 + p_r = \\ \nu\left[\frac{1}{r}(r(u^r)_r)_r + \frac{1}{B^2}(u^r)_{\xi\xi} - \frac{1}{r^2}u^r - \frac{2b^2B}{r^3}(u^\xi)_\xi\right],\end{aligned} \tag{6.2b}$$

$$0 \;=\; \nu\frac{2abB}{r^2}\left((u^r)_\xi - \left(Bu^\xi\right)_r\right), \tag{6.2c}$$

$$\begin{aligned}(u^\xi)_t + u^r(u^\xi)_r + \frac{1}{B}u^\xi(u^\xi)_\xi + \frac{b^2B^2}{r^3}u^r u^\xi + \frac{1}{B}p_\xi = \\ \nu\left[\frac{1}{r}(r(u^\xi)_r)_r + \frac{1}{B^2}(u^\xi)_{\xi\xi} + \frac{a^4B^4-1}{r^2}u^\xi + \frac{2b^2B}{r^3}(u^r)_\xi\right].\end{aligned} \tag{6.2d}$$

Note that the equation (6.2c) vanishes when $\nu ab = 0$, i.e., for inviscid flows, and for viscous flows with axial or planar symmetry. For other cases when the equation (6.2c) does not vanish, it imposes an additional differential constraint on the velocity components u^r, u^ξ. Such a restriction may lead to lack of solution existence for boundary value problems, and hence below only the inviscid case with $a, b \neq 0$ and both viscous and inviscid cases when $a = 0$ or $b = 0$ are considered.

6.1 General inviscid two-component helically invariant flow

In this section two-component helically invariant Euler flows satisfying (6.1) are considered. The three governing equations in primitive variables are given by (6.2a), (6.2b) and (6.2d), with $\nu = 0$. Employing first-order conservation law multipliers, one finds that the energy conservation law EP1 (4.1) is carried over without change; the conservation laws EP2 (4.2) and EP3 (4.3) collapse to one, given by:

$$\Theta = Bu^{\xi}, \quad \Phi^{r} = Bu^{r}u^{\xi}, \quad \Phi^{\xi} = B((u^{\xi})^{2} + p);$$

the conservation law EP4 (4.4) vanishes. No additional conservation laws arise in the above multiplier ansatz.

In the vorticity formulation, equations in primitive variables are appended with the definition of vorticity and the vorticity transport equations. For the two-component case, from the restriction (6.1) and the definition of vorticity components (2.32), it follows that

$$\omega^{r} = -\frac{1}{B}(u^{\eta})_{\xi} = -\frac{1}{B}\cdot 0 = 0, \tag{6.3a}$$

$$\omega^{\xi} = (u^{\eta})_{r} + \frac{a^{2}B^{2}}{r}u^{\eta} = 0 + \frac{a^{2}B^{2}}{r}\cdot 0 = 0. \tag{6.3b}$$

The remaining vorticity component ω^{η} is given by:

$$\omega^{\eta} = \frac{1}{B}(u^{r})_{\xi} - \frac{1}{r}\frac{\partial}{\partial r}(ru^{\xi}) + \frac{a^{2}B^{2}}{r}u^{\xi}. \tag{6.4}$$

The vorticity transport equations in $r-$ and $\xi-$directions vanish identically, and the remaining equation reads:

$$(\omega^{\eta})_{t} + \frac{1}{r}\frac{\partial}{\partial r}(ru^{r}\omega^{\eta}) + \frac{1}{B}\frac{\partial}{\partial \xi}(u^{\xi}\omega^{\eta}) - \frac{a^{2}B^{2}}{r}u^{r}\omega^{\eta} = 0. \tag{6.5}$$

Physically it is important to note that the reduction due to (6.1) gives rise to the elimination of the vortex stretching term in equation (2.33b). Hence, similar to the plane two-component case, equation (6.5) corresponds to pure helical vorticity convection. This vanishing of vortex stretching gives rise to an additional infinite family of conservation laws solely emerging from the vorticity equation (6.5), given by:

$$\Theta = T\left(\frac{B}{r}\omega^{\eta}\right), \quad \Phi^{r} = u^{r}T\left(\frac{B}{r}\omega^{\eta}\right), \quad \Phi^{\xi} = u^{\xi}T\left(\frac{B}{r}\omega^{\eta}\right), \tag{6.6}$$

where $T(\cdot)$ is an arbitrary function. The family (6.6) corresponds to an infinite family of material vorticity conservation laws given by:

$$\frac{d}{dt}\,T\left(\frac{B}{r}\omega^{\eta}\right) = 0. \tag{6.7}$$

The infinite-dimensional family of conservation laws (6.6) corresponds to a family of Casimir invariants (Bowman 2009); the case $T(q) = q^2$ may be referred to as "enstrophy conservation" in two-component helical flows.

Formulae (6.6) generalize the well-known plane two-component flow conservation laws listed below (formula (6.16)) and, moreover, give rise to an infinite family of conservation laws for axisymmetric flows which is also given (formula (6.22)).

Concerning the other conservation laws that were derived in chapter 4 for the three-component helically invariant Euler flows, it should be noted that in the two-component setting, the helicity conservation law EV1 (4.6) does not arise, since $h = \boldsymbol{u} \cdot \boldsymbol{\omega} \equiv 0$. Vorticity conservation laws EV2 (4.9) and EV6 (4.14) also vanish identically. The three conservation laws EV3 (4.10), EV4 (4.11) and EV5 (4.13) yield independent conserved quantities of the forms

$$\Theta^1 = \frac{Q(t)B}{r}\omega^\eta, \quad \Theta^2 = rB\omega^\eta.$$

6.2 The classical plane flow

For the two-component plane flow, as noted above, one can generally consider viscous flows, since the restriction (6.2c) vanishes. One now seeks zeroth-order conservation laws of z-invariant Navier-Stokes equations (2.17) in Cartesian coordinates, with an additionally imposed condition of vanishing velocity in the z-direction: $u^z = 0$. Where possible, we will extend the set of conservation laws admitted for inviscid flows, $\nu = 0$.

For the vorticity in planar flows, one has $\omega^x = \omega^y = 0$. The direct construction method is applied both to the system (2.17) in primitive variables, and to the vorticity system which involves the equations

$$\omega^z + (u^x)_y - (u^y)_x = 0, \tag{6.8a}$$

$$(\omega^z)_t + u^x(\omega^z)_x + u^y(\omega^z)_y = \nu\left[(\omega^z)_{xx} + (\omega^z)_{yy}\right]. \tag{6.8b}$$

It should be noted, that in the current section, since all scaling factors in Cartesian coordinates are ones, local conservation laws (3.8) have the form

$$\frac{\partial \Theta}{\partial t} + \frac{\partial \Phi^x}{\partial x} + \frac{\partial \Phi^y}{\partial y} = 0.$$

For the general viscous case, the following conservation laws arise: one has the conservation of angular momentum in the $z-$ direction, given by:

$$\begin{aligned} \Theta &= yu^x - xu^y, \\ \Phi^x &= y(u^x)^2 - xu^xu^y + yp + \nu(x(u^y)_x - y(u^x)_x - u^y), \\ \Phi^y &= yu^xu^y - x(u^y)^2 - xp + \nu(x(u^y)_y - y(u^x)_y + u^x). \end{aligned} \tag{6.9}$$

Note that for a general helical setting it only holds for inviscid flows, cf. (4.3). Further, even in the viscous setting, one readily computes two families of conservation laws sometimes referred to as the "center of mass theorem" (Caviglia & Morro 1989) in $x-$ and $y-$directions:

$$\Theta = f_1(t)u^x, \qquad \begin{aligned} \Phi^x &= f_1(t)\left((u^x)^2 + p - \nu(u^x)_x)\right) - x f_1'(t)u^x, \\ \Phi^y &= f_1(t)\left(u^x u^y - \nu(u^x)_y\right) - x f_1'(t)u^y, \end{aligned} \tag{6.10}$$

$$\Theta = f_2(t)u^y, \qquad \begin{aligned} \Phi^x &= f_2(t)\left(u^x u^y - \nu(u^y)_x\right) - y f_2'(t)u^x, \\ \Phi^y &= f_2(t)\left((u^y)^2 + p - \nu(u^y)_y\right) - y f_2'(t)u^y, \end{aligned} \tag{6.11}$$

where $f_1(t), f_2(t)$ are arbitrary functions. The term "center of mass theorem" is more appropriate for compressible gas rather than a constant-density unbounded fluid at rest at infinity. In the setting of the current contribution, it seems more appropriate to refer to formulae (6.10) and (6.11) as the "generalized momentum" conservation laws. If A is the two-dimensional domain occupied by the fluid, with no-leak boundary conditions $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on the boundary ∂A, then from (6.10), one has the balance law

$$\frac{d}{dt} \iint_A f(t)u^x dA = \int_{\partial A} f(t) \left[(p,0) \cdot \boldsymbol{n} - \nu(\nabla u^x) \cdot \boldsymbol{n}\right] \, d\ell$$

for the generalized $x-$momentum $f(t)u^x$, where the $\boldsymbol{n}$ is the unit exterior normal to ∂A; a similar law arising from (6.11) holds for the $y-$direction.

For inviscid flows, conservation laws (6.10) and (6.11) are known to hold, in Cartesian coordinates, for the general 3D Euler equations (Caviglia & Morro 1987, Caviglia & Morro 1989). For viscous flows, these conservation laws are new, to the best of authors knowledge. Note that these families do not arise for the general helical Euler or Navier-Stokes system (cf. chapter 4).

Using the direct method with zeroth-order multiplies for the vorticity formulation, i.e., taking into account equations (6.8), one can additionally derive the following conservation laws that hold for viscous two-component planar flows:

$$\begin{aligned} \Theta &= \frac{x^2+y^2}{2}\omega^z, \\ \Phi^x &= \frac{1}{2}\left(u^x\omega^z\left(x^2+y^2\right) + y\left((u^x)^2 - (u^y)^2\right)\right) - x u^x u^y \\ &\quad + \nu\left(x\omega^z - \tfrac{1}{2}(x^2+y^2)(\omega^z)_x - 2u^y\right), \\ \Phi^y &= \frac{1}{2}\left(u^y\omega^z\left(x^2+y^2\right) + x\left((u^x)^2 - (u^y)^2\right)\right) + y u^x u^y \\ &\quad + \nu\left(y\omega^z - \tfrac{1}{2}(x^2+y^2)(\omega^z)_y + 2u^x\right); \end{aligned} \tag{6.12}$$

$$\begin{aligned} \Theta &= f_3(t)\omega^z, \\ \Phi^x &= f_3(t)\left(u^x\omega^z - \nu(\omega^z)_x\right) - f_3'(t)u^y, \\ \Phi^y &= f_3(t)\left(u^y\omega^z - \nu(\omega^z)_y\right) + f_3'(t)u^x; \end{aligned} \tag{6.13}$$

$$\begin{aligned} \Theta &= f_4(t)x\omega^z, \\ \Phi^x &= f_4(t)\left(xu^x\omega^z - u^x u^y + \nu(\omega^z - x(\omega^z)_x)\right) + f_4'(t)\left(yu^x - xu^y\right), \\ \Phi^y &= f_4(t)\left(xu^y\omega^z + \tfrac{1}{2}(u^x)^2 - \tfrac{1}{2}(u^y)^2 - \nu x(\omega^z)_y\right) + f_4'(t)\left(xu^x + yu^y\right); \end{aligned} \tag{6.14}$$

$$\begin{aligned}
\Theta &= f_5(t) y \omega^z, \\
\Phi^x &= f_5(t)\left(y u^x \omega^z + \tfrac{1}{2}(u^x)^2 - \tfrac{1}{2}(u^y)^2 - \nu y (\omega^z)_x\right) - f_5'(t)\left(x u^x + y u^y\right), \\
\Phi^y &= f_5(t)\left(y u^y \omega^z + u^x u^y + \nu(\omega^z - x(\omega^z)_y)\right) + f_5'(t)\left(y u^x - x u^y\right),
\end{aligned} \tag{6.15}$$

where $f_3(t), f_4(t)$ and $f_5(t)$ are arbitrary functions.

The conservation law (6.12) and the particular cases of the families (6.13), (6.14) and (6.15) for $f_3(t), f_4(t), f_5(t) = const.$ are known in the literature for the inviscid case (e.g., Batchelor (2000)).

To the best of the authors knowledge, the conservation laws (6.12)–(6.15) have not been known in the viscous setting and for general forms of $f_3(t), f_4(t), f_5(t)$.

For the constant values of the arbitrary functions, formula (6.13) describes the conservation of the $z-$component of the vorticity vector. In particular, for an unbounded fluid at rest at infinity,

$$\frac{d}{dt}\iint \omega^z dA = 0.$$

Similarly, formulae (6.14) and (6.15) represent the conservation of the first two moments:

$$\frac{d}{dt}\iint x\omega^z dA = \frac{d}{dt}\iint y\omega^z dA = 0,$$

and the quantities

$$X = \frac{\iint x\omega^z dA}{\iint \omega^z dA}, \quad Y = \frac{\iint y\omega^z dA}{\iint \omega^z dA}$$

are naturally interpreted as the coordinates of the "center of vorticity" (Batchelor 2000).

Taking non-constant $f_3(t), f_4(t), f_5(t)$ in formulae (6.13), (6.14) and (6.15) corresponds to non-homogeneous time rescaling in the evolution of $\int \omega^z dA$, $\int x\omega^z dA$, and $\int y\omega^z dA$ in boundary value problems for which the corresponding integrals are not conserved.

The conservation law (6.12) is a second radial moment of ω^z, and is related to the vorticity dispersion, as discussed by Batchelor (2000).

For inviscid two-component flows, one additionally has a well-known family of vorticity conservation laws (Bowman 2009) given by:

$$\Theta = N(\omega^z), \quad \Phi^x = u^x N(\omega^z), \quad \Phi^y = u^y N(\omega^z), \tag{6.16}$$

which are readily obtained as a reduction of our general formulae (6.6) onto the plane case. The conservation laws (6.16) are clearly of the material form:

$$\frac{d}{dt}N(\omega^z) = 0.$$

The case when $N(\omega^z) = (\omega^z)^2$ corresponds to the conservation of enstrophy. In general, conserved quantities $N(\omega^z)$ are referred to as Casimirs by Bowman (2009). The family (6.16) does not admit a viscous extension.

6.3 The axisymmetric case

In this section one seeks local conservation laws of two-component axisymmetric flows, i.e., flows satisfying the following condition:

$$u^{\varphi} = 0 \tag{6.17}$$

in both the viscous and the inviscid setting, and compare them with the conservation laws obtained for general helically invariant viscous flows (chapter 5) and for the two-component helically invariant inviscid flows (subsection 6.1).

For flows satisfying (6.17), one has $\omega^r = \omega^z = 0$, and the remaining vorticity equations read

$$\omega^{\varphi} + (u^z)_r - (u^r)_z = 0, \tag{6.18a}$$

$$(\omega^{\varphi})_t + u^r\left((\omega^{\varphi})_r - \frac{1}{r}\omega^{\varphi}\right) + u^z(\omega^{\varphi})_z = \nu\left[(\omega^{\varphi})_{rr} + \frac{1}{r}(\omega^{\varphi})_r - \frac{1}{r^2}\omega^{\varphi} + (\omega^{\varphi})_{zz}\right]. \tag{6.18b}$$

Similarly to the planar two-component case, for axisymmetric flows, various previously known conservation laws carry over or vanish, but also new conservation laws arise that have no direct counterpart for the general helically symmetric setting. In the current section, the conservation laws are listed in cylindrical coordinates, and have the form:

$$\frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{\partial \Phi^z}{\partial z} = 0.$$

Starting from the general rotationally symmetric Navier-Stokes equations in primitive variables (2.15a)-(2.15c), one readily obtains an infinite-dimensional set of conservation laws given by the density and the fluxes:

$$\Theta = g_1(t)u^z, \qquad \begin{aligned} \Phi^r &= g_1(t)\left(u^r u^z - \nu(u^z)_r)\right) - z g_1'(t)u^r, \\ \Phi^z &= g_1(t)\left((u^z)^2 + p - \nu(u^z)_z\right) - z g_1'(t)u^z, \end{aligned} \tag{6.19}$$

holding for an arbitrary function $g_1(t)$. The conservation law (6.19) corresponds to the conservation of the "generalized momentum" in the $z-$direction, and similarly to the conservation laws (6.10) and (6.11) for plane flows, holds in both viscous and inviscid settings.

In the vorticity formulation, taking into account equations (6.18) and using zeroth-order multiplies in the direct method, one additionally finds two families of conservation laws, given by:

$$\begin{aligned} \Theta &= \frac{1}{r}g_2(t)\omega^{\varphi}, \\ \Phi^r &= \frac{1}{r}\left(g_2(t)\left[u^r\omega^{\varphi} - \nu\left((\omega^{\varphi})_r + \frac{1}{r}\omega^{\varphi}\right)\right] + g_2'(t)u^z\right), \\ \Phi^z &= \frac{1}{r}\left(g_2(t)\left[u^z\omega^{\varphi} - \nu(\omega^{\varphi})_z\right] - g_2'(t)u^r\right), \end{aligned} \tag{6.20}$$

and

$$\begin{aligned}
\Theta &= g_3(t) r\omega^\varphi, \\
\Phi^r &= g_3(t)\left[ru^r\omega^\varphi + 2u^r u^z + \nu\left(\omega^\varphi - r(\omega^\varphi)_r\right)\right] + g_3'(t)(ru^z - 2zu^r), \\
\Phi^z &= g_3(t)\left[(u^z)^2 - (u^r)^2 + ru^z\omega^\varphi - \nu r(\omega^\varphi)_z\right] - g_3'(t)(ru^r + 2zu^z),
\end{aligned} \tag{6.21}$$

for arbitrary $g_2(t)$ and $g_3(t)$. The family (6.20) is an axially symmetric restriction of the conservation laws NSV1 (5.3) found above. The family (6.21) is new; it is specific for two-component axially symmetric flows, and does not hold in a general helical setting. The families (6.20) and (6.21) describe the conservation of two different generalized $r-$moments of the fluid vorticity. As discussed in the remark after the formula (4.11), for some flows, the conservation law (6.21) may also be interpreted as generalized conservation of linear momentum in the z-direction.

For inviscid flows, the set of admitted conservation laws is extended by a family of Casimir invariants (6.7), which in an axially symmetric setting take the form

$$\Theta = S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^r = u^r S\left(\frac{1}{r}\omega^\varphi\right), \quad \Phi^z = u^z S\left(\frac{1}{r}\omega^\varphi\right), \tag{6.22}$$

where $S(\cdot)$ is an arbitrary function of its argument. The conservation laws (6.22) are material conservation laws:

$$\frac{d}{dt} S\left(\frac{1}{r}\omega^\varphi\right) = 0.$$

7 DNS of flows with helical symmetry

In the present chapter the numerical code HELIX will be shortly described. Further, local conservation laws from previous chapters (chapters 4 and 5) will be integrated in order to obtain global conservation laws. It can be expected, that the conserved quantities for Navier-Stokes equations (chapter 5) will be preserved by the numerical code. In case of inviscid flow (chapter 4) it can not be expected to see the conserved properties of the flow due to the presence of the viscosity in the code. Nevertheless, one can think of the conservation in an approximative sense: for the simulations with high Reynolds number the effect of the viscosity becomes small, the Navier-Stokes equations approximate the Euler equations and the conservation laws for Euler equations should be confirmed in this approximative sense.
There are several different opportunities, which can be used for the investigation of the flow dynamics. For the purpose of this thesis a **D**irect **N**umerical **S**imulation (DNS) is the appropriate choice. The Navier-Stokes equations are solved for all kind of structures, additional equations/modells are not necessary. The high precision of the results of the simulation is the big advantage of this method. One has to mention that the applicability of DNS is limited, e.g. simulations of complex systems with high Reynolds number are not feasible due to the high computational time.

7.1 Description of the code HELIX

The DNS code HELIX, which was used in this work, was developed by Ivan Delbende, Maurice Rossi and Olivier Daube. A detailed description can be found in Delbende et al. (2012). This code is adapted from an older code, which integrates the two-dimensional Navier-Stokes equations in velocity-vorticity formulation (Daube 1992). On the next pages, a short overview of the numerics, described in Delbende et al. (2012), will be given. Please note that the original nomenclature of Delbende et al. is slightly different to the nomenclature which is used in the present work. In order to avoid a possible confusion and to stay consistent one will stick to the notation of this dissertation.

The velocity field for incompressible helically symmetric flows can be expressed with two scalar fields: velocity component $u^\eta\,(t,r,\xi)$ along the invariant direction η and the stream function $\Psi\,(t,r,\xi)$:

$$\boldsymbol{u} = u^\eta \boldsymbol{e}_\eta + \frac{B}{r}\nabla\Psi \times \boldsymbol{e}_\eta. \tag{7.1}$$

The components u^r and u^ξ can be obtained from

$$u^r = -\frac{1}{r}\Psi_\xi, \quad u^\xi = \frac{B}{r}\Psi_r \tag{7.2}$$

and

$$\nabla\Psi = \Psi_r \boldsymbol{e}_r + \frac{1}{B}\Psi_\xi \boldsymbol{e}_\xi.$$

The vorticity vector can be written as:

$$\boldsymbol{\omega} = \omega^\eta \boldsymbol{e}_\eta + \frac{B}{r}\nabla\left(\frac{r}{B}u^\eta\right) \times \boldsymbol{e}_\eta. \tag{7.3}$$

The components ω^r and ω^ξ are given by

$$\omega^r = -\frac{1}{r}\left(\frac{r}{B}u^\eta\right)_\xi, \quad \omega^\xi = \frac{B}{r}\left(\frac{r}{B}u^\eta\right)_r, \tag{7.4}$$

whereas the vorticity component ω^η can be expressed by use of the stream function Ψ and the velocity component u^η:

$$\omega^\eta = -\mathbb{L}\Psi - \frac{2abB^2}{r^2}u^\eta. \tag{7.5}$$

In this equation $\mathbb{L}$ is a linear operator defined by:

$$\mathbb{L}(\cdot) = \frac{1}{b^2 B}\left(\frac{b^2 B^2}{r}(\cdot)_r\right)_r + \frac{1}{rB}(\cdot)_{\xi\xi}. \tag{7.6}$$

From the equations (7.1) and (7.3) one can see, that the velocity and the vorticity fields are described by only two scalar fields $u^\eta(t, r, \xi)$ and $\omega^\eta(t, r, \xi)$ and the stream function $\Psi(t, r, \xi)$ is connected to these variables by equation (7.5). The dynamics of the flow can then be described by two equations: the momentum equation (2.13c) for u^η and the transport equation (2.33b) for ω^η. These two equations can be written in a slightly different way. Therefor, we consider the incompressible Navier-Stokes equations in vector notation:

$$\boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \frac{\boldsymbol{u}^2}{2}\right) - \nu\nabla \times \boldsymbol{\omega}. \tag{7.7}$$

The projection of the equation (7.7) in the direction of $\boldsymbol{e}_\eta$ leads to the equation for u^η:

$$(u^\eta)_t + NL_u = VT_u. \tag{7.8}$$

Taking a curl of the equation (7.7) and then projecting it in $\eta-$direction yield the equation for ω^η:

$$(\omega^\eta)_t + NL_\omega = VT_\omega. \tag{7.9}$$

The viscous terms in the equations (7.8) and (7.9) are the following:

$$VT_u = -\nu \boldsymbol{e}_\eta \cdot [\nabla \times \boldsymbol{\omega}] = \nu \left[\mathbb{L}\left(\frac{r}{B} u^\eta\right) + \frac{2abB^2}{r^2}\omega^\eta \right], \tag{7.10}$$

$$VT_\omega = -\nu \boldsymbol{e}_\eta \cdot \nabla \times [\nabla \times \boldsymbol{\omega}] = \nu \left[\mathbb{L}\left(\frac{r}{B}\omega^\eta\right) - \left(\frac{2abB^2}{r^2}\right)^2 \omega^\eta \right] - \nu \frac{2abB^2}{r^2} \mathbb{L}\left(\frac{r}{B}u^\eta\right). \tag{7.11}$$

The nonlinear terms are given by:

$$NL_u = \boldsymbol{e}_\eta \cdot [\boldsymbol{\omega} \times \boldsymbol{u}], \tag{7.12}$$

$$NL_\omega = \boldsymbol{e}_\eta \cdot \nabla \times [\boldsymbol{\omega} \times \boldsymbol{u}]. \tag{7.13}$$

Summing up, the complete set of equations is defined by equation (7.8) for u^η, equation (7.9) for ω^η, equation (7.5) for Ψ, which connects these three variables, and the boundary conditions, which will be defined later. This $\Psi - \omega^\eta - u^\eta$ formulation turns out to be a generalisation of the standard two-dimensional $\Psi - \omega$ method.

Because the velocity $u^\eta(t, r, \xi)$ and the vorticity $\omega^\eta(t, r, \xi)$ are $2\pi-$periodic with respect to the variable ξ, these fields can be expressed as Fourier series along ξ. One introduces the complex modes $u^{\eta(m)}(t, r)$ and $\omega^{\eta(m)}(t, r)$ via

$$u^\eta(t, r, \xi) = \sum_{m=0}^{q} u^{\eta(m)}(t, r) e^{\mathrm{i}m\xi} \tag{7.14}$$

and

$$\omega^\eta(t, r, \xi) = \sum_{m=0}^{q} \omega^{\eta(m)}(t, r) e^{\mathrm{i}m\xi}, \tag{7.15}$$

for which the equation (7.8) and (7.9) can be recast for each Fourier mode m. The modes $u^{\eta(m)}$ and $\omega^{\eta(m)}, m \neq 0$ yield all the other quantities: the equation (7.5) leads to the value $\Psi^{(m)}$, and from $\Psi^{(m)}$ one obtains $u^{r(m)}$ and $u^{\xi(m)}$ through

$$u^{r(m)}(t, r) = -\frac{\mathrm{i}m}{r}\Psi^{(m)}, \quad u^{\xi(m)}(t, r) = \frac{B}{r}\Psi_r^{(m)}. \tag{7.16}$$

The viscous terms $VT_u^{(m)}$ and $VT_\omega^{(m)}$ in the recast dynamical equations for $u^{\eta(m)}$ and $\omega^{\eta(m)}$ can be directly obtained from VT_u and VT_ω because of the linearity of these terms. For the nonlinear terms one has to evaluate first NL_u and NL_ω in physical space and then Fourier-transform to yield the various coefficients $NL_u^{(m)}$ and $NL_\omega^{(m)}$ (pseudo-spectral formulation). Subsequently we describe the adapted time stepping scheme. For this, let $G^{(m)}$ be defined as follows:

$$G^{(m)} = \left(u^{\eta(m)}, \omega^{\eta(m)}\right).$$

The value $G_{n+1}^{(m)}$ at time step $n+1$ is obtained by:

$$\frac{3G_{n+1}^{(m)} - 4G_{n}^{(m)} + G_{n-1}^{(m)}}{2\Delta t} + \left(2NL_{n}^{(m)} - NL_{n-1}^{(m)}\right) = VT_{n+1}^{(m)} \tag{7.17}$$

with

$$NL^{(m)} = \left(NL_{u}^{(m)}, NL_{\omega}^{(m)}\right), \quad VT^{(m)} = \left(VT_{u}^{(m)}, VT_{\omega}^{(m)}\right).$$

For the temporal derivatives a second order backward discretization is used, the nonlinear terms are modeled explicit through second order Adams-Bashforth extrapolation, the viscous terms are discretized implicit.

7.1.1 Boundary conditions

Close to the outer boundary $r = R_{ext}$ the flow is assumed to be potential, that means

$$\omega^{\eta(m)}\left(r = R_{ext}\right) = 0. \tag{7.18}$$

Further, ω^r and ω^ξ tend to zero outside the vorticity region, this means $u^{\eta(m)}$ decreases for large r:

$$u^{\eta(m)}\left(r = R_{ext}\right) = 0. \tag{7.19}$$

In addition, asymptotic behaviour of the stream function $\Psi^{(m)}$ at large r is used. This assumption is valid because the vorticity region is included in the domain with the radius R_{ext}.

Because of the regularity conditions at $r = 0$ one has the following boundary conditions at the inner boundary:

$$\Psi^{(m)}\left(r = 0\right) = 0, \quad \omega^{\eta(m)}\left(r = 0\right) = 0, \quad u^{\eta(m)}\left(r = 0\right) = 0. \tag{7.20}$$

7.1.2 Initial conditions

There are several types of initial conditions, which can be used for the DNS of flows with helical symmetry, the choice of the appropriate initial condition depends on the purpose. In this work two slightly different initial conditions are used. The first type corresponds to a regular field of bean-shaped vortices. These are placed on a ring of a fixed radius $R_0 = 1$. The initial distribution of ω^η is assumed to reproduce the shape of a helical Gaussian vortex, multiplied by a fixed number. For the initial distribution of the velocity u^η one can assume an exponential profile as well, multiplied by the same fixed number. The second type of used initial conditions is a random field of bean-shaped vortices. Now, the vortices are placed on a disc of a radius $R_0 = 1$. The randomness appears in the initial distribution of the vortices and in the initial value of the vorticity as well as velocity in the invariant direction η. The random position of the vortices is implemented via uniform probability in azimuthal direction and nonuniform distribution along the radial direction. The initial distribution of ω^η is assumed to reproduce the shape of a helical Gaussian vortex with core size A situated at (R_0, φ_0) in the (r, φ)-plane, multiplied by a random number N_{rnd}:

$$\omega^\eta = N_{rnd}\text{exp}\left(-\frac{(r-R_0)^2}{A^2} + \frac{B(R_0)^2 b^2}{R_0^2}\frac{r^2(\varphi-\varphi_0)^2}{A^2}\right),$$

For the initial distribution of the velocity u^η one can assume an exponential profile as well, multiplied by the same random number.

7.2 From local to global: integration of local conservation laws

The conservation laws from previous chapters are written in a local form:

$$\frac{\partial\Theta}{\partial t} + \nabla\cdot\Phi = \frac{\partial\Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial\Phi^\xi}{\partial\xi} = 0. \quad (7.21)$$

To obtain the global invariants from the local invariants one has to integrate equation (7.21) on a cylinder with the extent $[0, h]$ and radius R_{ext} along the z-axis (figure 7.1):

$$\begin{aligned}\frac{\partial}{\partial t}\int\int\int_V \Theta dV &= -\int_V \nabla\cdot\Phi dV \\ &= -\oint_{\partial V} \boldsymbol{\Phi}\cdot\boldsymbol{n} dS \\ &= -\int_{S_L}\Phi^r dS - \int_{S_T}\Phi^z dS + \int_{S_B}\Phi^z dS.\end{aligned} \quad (7.22)$$

The flux Φ^z can be rewritten as follows:

$$\Phi^z = B\left(-\frac{b}{r}\Phi^\eta + a\Phi^\xi\right).$$

Because all quantities in the present case are independent of η, the derivatives $\frac{\partial}{\partial\eta}$ are zero. For the spatial fluxes of the conservation law it means that only the fluxes Φ^r and Φ^ξ have to be taken into account, the flux Φ^η does not exist. In case of η-dependence one has to assume the periodicity for Φ^η.

The flux Φ^z then reduces to:

$$\Phi^z = aB\Phi^\xi$$

and the expression (7.22) becomes:

$$\frac{\partial}{\partial t}\int\int\int_V \Theta dV = -\int_{S_L}\Phi^r dS + \int_{S_T} aB\Phi^\xi dS - \int_{S_B} aB\Phi^\xi dS. \quad (7.23)$$

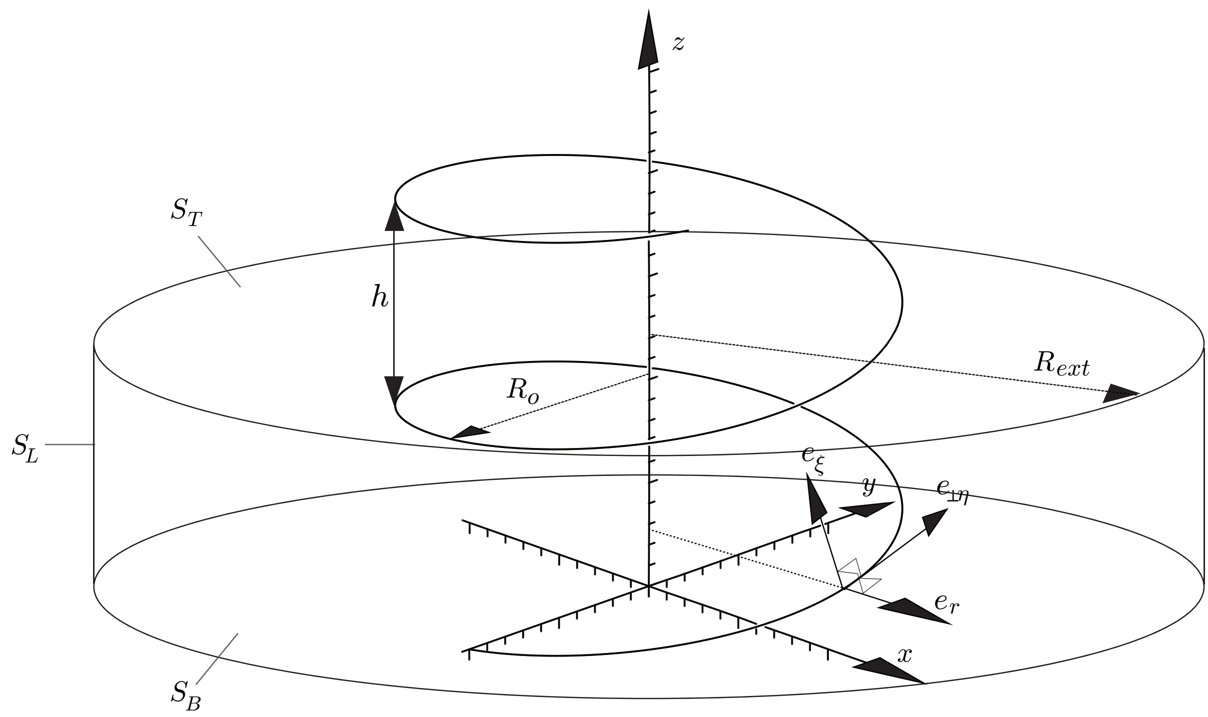


Figure 7.1: Integration domain.

The flux Φ^ξ is 2π-periodic and for (7.23) follows:

$$\frac{\partial}{\partial t}\int\int\int_V \Theta dV = -\int_{S_L} \Phi^r dS. \tag{7.24}$$

This can be written as

$$\frac{\partial}{\partial t}\int\int_S \Theta r dr d\varphi = -\oint_{\partial S} \Phi^r r d\varphi \tag{7.25}$$

where S is a disk e.g. at $z = 0$. Let denote $\int\int_S r\Theta dr d\varphi$ by Q, then the expression (7.25) becomes:

$$\frac{\partial Q}{\partial t} = -\oint_{\partial S} \Phi^r r d\varphi. \tag{7.26}$$

The quantity Q as well as the flux across the boundary close to $r = R_{ext}$ given by the right hand side of the equation (7.26) were evaluated for several simulations. It could be confirmed, that the obtained values across the boundary ∂S at $r = 5.5$ were sufficiently small (appendix A, Table A.1 and A.2). In order to obtain a global conservation law the quantity Q should become constant. In the next section the plots of the time evolution of the quantity $Q_{(\cdot)}$ will be presented. The lower index of $Q_{(\cdot)}$ will indicate the corresponding conservation law from chapter 4 and chapter 5.
It should be mentioned, that all simulations, which were done for the present dissertation, are performed under the assumption, that the vorticity is concentrated inside a cylinder of the radius $R_0 < R_{ext}$ and exponentially small outside the cylinder. Fur-

IC	Number of vortices	Reynolds number Re	reduced helical pitch X_L
regular IC	3	$1000; 5000$	$0.5; 1; 2$
	15	$1000; 5000$	$0.5; 1; 2$
random IC	6	$1000; 5000$	$0.5; 1; 2$
	30	$1000; 5000$	$0.5; 1; 2$

Table 7.1: Overview of the simulations.

ther, only the helically symmetric case is considered, i.e. the parameters a and b are not arbitrary, but set to $a = 1, b = -\frac{h}{2\pi}$.

7.3 Numerical results

In this section the results of some selected simulations will be presented (an overview of the parameters for these simulations is given in Table 7.1). Contour plots of the vorticity component ω^η for $t = 0$ (the time evolution of the vorticity component for the whole duration of the simulation can be found in the appendix A), as well as the plots of the evolution of integral quantities $Q_{(\cdot)}$ will be shown. The simulations are done for both types of initial conditions (IC): regular and random type (subsection 7.1.2). The number of vortices varies from three to fifteen for the regular case, for the random case simulations with six and thirty vortices were performed. Two different Reynolds numbers were chosen. Further, helical pitch h was varied as well. Let X_L denote the reduced helical pitch ($X_L = \frac{h}{2\pi}$), then e.g.

$$X_{L,\mathrm{C3}} = 2X_{L,\mathrm{C2}} = 4X_{L,\mathrm{C1}}.$$

Here C3, C2 and C1 are three simulations, which differ only in the reduced helical pitch, the remaining parameters were kept the same for these three simulations. The reason for the variation of helical pith is the following: by setting $X_L = 1$ one obtains a regular helically symmetric case (regular in a geometrical sense). Reduction of X_L by a half ($X_L = 0.5$) leads to a flow geometry, which approaches the case of rotationally symmetric flow. In contrast, doubling of X_L ($X_L = 2$) is corresponding to the case of plane flow.

Let N_r denote the number of grid points in radial direction, the number of grid points in azimuthal direction is N_{th}. Because of the initial placement of the vortices at a ring/disk of a radius $r \leq R_0 = 1$, one chooses the following ratio:

$$\frac{R_{ext}}{N_r} = \frac{2\pi r}{N_{th}}\Big|_{r=1}. \tag{7.27}$$

To obtain a regular cell ($N_r = N_{th}$) at $r = 1$ one sets $R_{ext} = 6$:

$$\frac{R_{ext}}{N_r} = \frac{2\pi 1}{N_{th}} \quad \Leftrightarrow \quad \frac{6}{N_r} \approx \frac{2\pi}{N_{th}}. \tag{7.28}$$

It should be noted, that for reasons of better comparability, the plots of the time evolution of the quantities $Q_{(\cdot)}$ are not showing the absolute values, but the normalized ones. The normalization was done with the corresponding maximum value of each $Q_{(\cdot)}$. The solid line represents the result of the simulation with the smallest pitch ($X_L = 0.5$), the dashed line corresponds to the pitch $X_L = 1$, the dashed-dotted line shows the result of the simulation with the largest pitch ($X_L = 2$). Further, conservation laws, which are containing arbitrary functions, were simulated for different choices of these functions. In case of conservation law EP4 (4.4):

$$\begin{aligned}
\text{"EP41"} \quad &\widehat{=} \quad F\left(\frac{r}{B}u^{\eta}\right) = \frac{r}{B}u^{\eta}, \\
\text{"EP42"} \quad &\widehat{=} \quad F\left(\frac{r}{B}u^{\eta}\right) = \frac{r^2}{B^2}\left(u^{\eta}\right)^2 \quad \text{etc.}
\end{aligned}$$

For the conservation law EV2 (4.9) one has:

$$\begin{aligned}
\text{"EV2H0"} \quad &\widehat{=} \quad H\left(\frac{r}{B}u^{\eta}\right) = 1, \\
\text{"EV2H1"} \quad &\widehat{=} \quad H\left(\frac{r}{B}u^{\eta}\right) = \frac{r}{B}u^{\eta}, \\
\text{"EV2H2"} \quad &\widehat{=} \quad H\left(\frac{r}{B}u^{\eta}\right) = \frac{r^2}{B^2}\left(u^{\eta}\right)^2 \quad \text{etc.}
\end{aligned}$$

In subsection 7.3.1 the plots for all these choises of arbitrary functions will be shown. It will be seen, that the results resemble each other in appearance, therefore in subsection 7.3.2 these quantities will be shown only for one representative case. Conservation law EV3 (4.10) and accordingly NSV1 (5.3) could be only simulated for $Q(t) = 1$.

7.3.1 Regular IC

The parameters for the simulations in case of $Re = 1000$ (CSI1 - CSI3) are the following:

$$N_r = N_{th} = 372, \quad \Delta r = 0.01612, \quad \Delta t = 0.0005, \quad N_{It} = 10000,$$

$$R_{ext} = 6.0, \quad R_0 = 1.0, \quad A_0 = 0.1.$$

N_r and N_{th} correspond to the number of grid points in radial and azimuthal direction respectively, Δr stands for the cell size, Δt is the time step. N_{It} is the number of iterations, Re is the Reynolds number. Geometrical parameters are the radius of the computational domain R_{ext}, helix radius R_0 and the vortex core size A_0.
For the case $Re = 5000$ (CSI500 - CSI700) one has:

$$N_r = N_{th} = 1024, \quad \Delta r = 0.0059, \quad \Delta t = 0.0002, \quad N_{It} = 25000,$$

$$R_{ext} = 6.0, \quad R_0 = 1.0, \quad A_0 = 0.1.$$

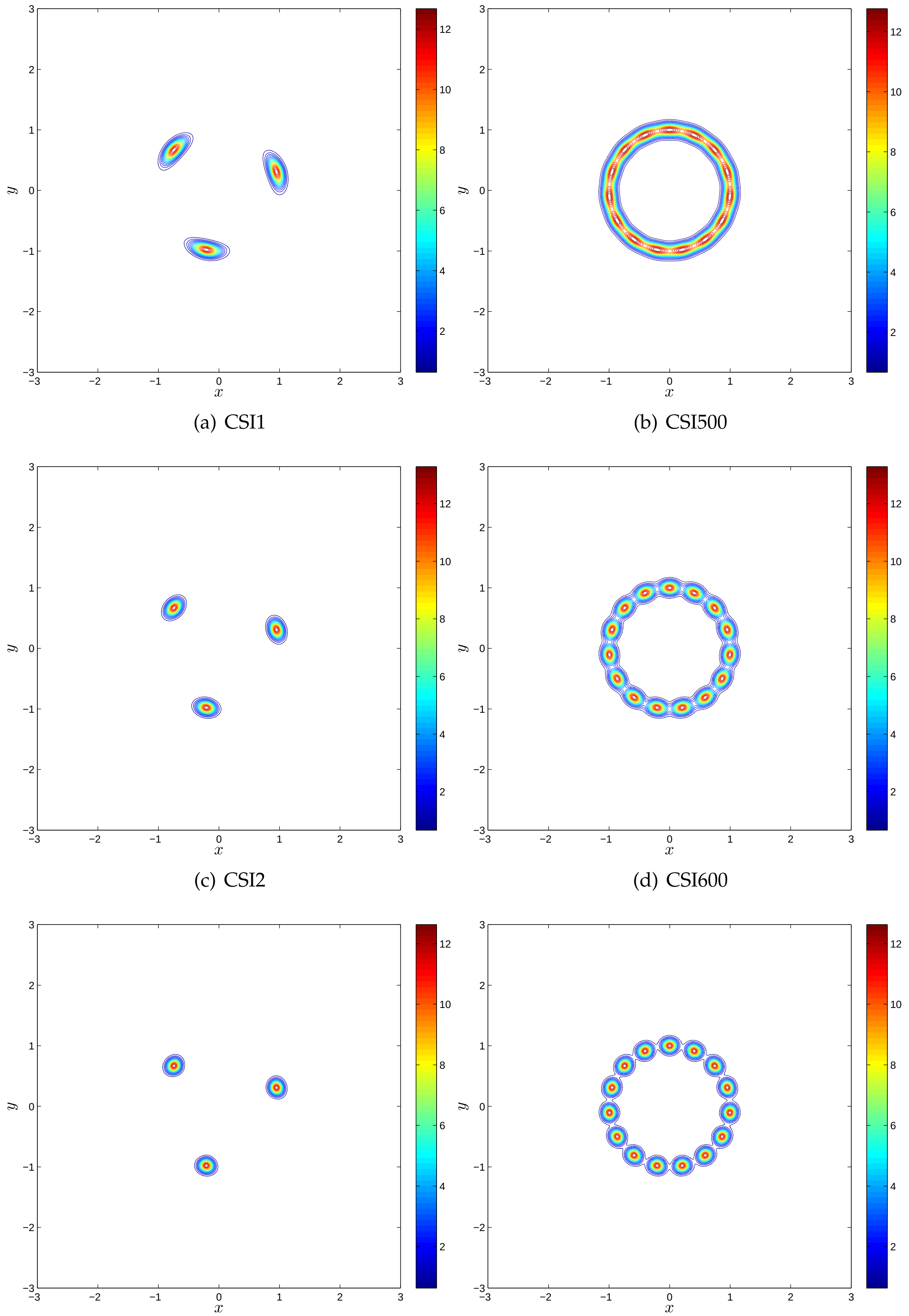
The reduced pitch X_L and the number of vortices N_{vort} were varied as follows:

$$\text{CSI1}: \quad X_L = 0.5, \quad N_{vort} = 3; \qquad \text{CSI500}: \quad X_L = 0.5, \quad N_{vort} = 15;$$

$$\text{CSI2}: \quad X_L = 1.0, \quad N_{vort} = 3; \qquad \text{CSI600}: \quad X_L = 1.0, \quad N_{vort} = 15;$$

$$\text{CSI3}: \quad X_L = 2.0, \quad N_{vort} = 3; \qquad \text{CSI700}: \quad X_L = 2.0, \quad N_{vort} = 15.$$

As can be seen from the following plots, the conservation of the momentum (EP2 and NSP1), of the angular momentum (EP3), of the generalized momentum/angular momentum for $n = 1$ (EP41, NSP2), and of the vorticity (EV3, NSV1 and EV4, NSV2) is matched by the simulations CSI1, CSI2, CSI3. The conservation of the vorticity in EV5 and NSV3 could be confirmed by the simulations CSI2 and CSI3. In case of the simulation CSI1 the quantity EV5 and accordingly NSV3 decreases in a linear manner. In case of the generalized momenta/angular momenta (EP42 - EP45) and the family of generalized helicity conservation laws (EV2H0 - EV2H4) the gradient tends faster to zero for increasing power of the quantity for all three simulations. The remaining quantities become constant after approximately thirty percent of the simulation time. Increasing the number of vortices from three to fifteen and keeping the remaining parameters unchanged leads to the following results: one obtains the same behaviour as for the simulations CSI1, CSI2, CSI3. It can be noticed, that the simulations for the pithes $X_L = 1$ and $X_L = 2$ display the conservation property in a slightly better manner, than the simulation for the smallest pitch ($X_L = 0.5$). In general, the analytical results are reproduced by the code very well, even the quantities, which only hold for inviscid flows, are preserved after thirty percent of the simulation time.
Increasing the Reynolds number from $Re = 1000$ to $Re = 5000$ leads to the following results: for the small amount of vortices the results are conform with those for the the case $Re = 1000$; for the simulations CSI500 - CSI700 (fifteen vortices) the conclusions from the previous passage can be transferred as well. Comparing the cases $Re = 1000$ and $Re = 5000$ it can be stated, that the curves become less smooth with increasing Reynolds number. Again, the simulations for the pithes $X_L = 1$ and $X_L = 2$ display the conservation property in a better manner, than the simulation for the smallest pitch ($X_L = 0.5$), but compared to the case of $Re = 1000$ the results for the smallest pitch become worse in the sense of becoming constant.



(a) CSI1 (b) CSI500 (c) CSI2 (d) CSI600 (e) CSI3 (f) CSI700

Figure 7.2: Contour plots of the vorticity component ω^η for different simulations at $t = 0$, $z = 0$.

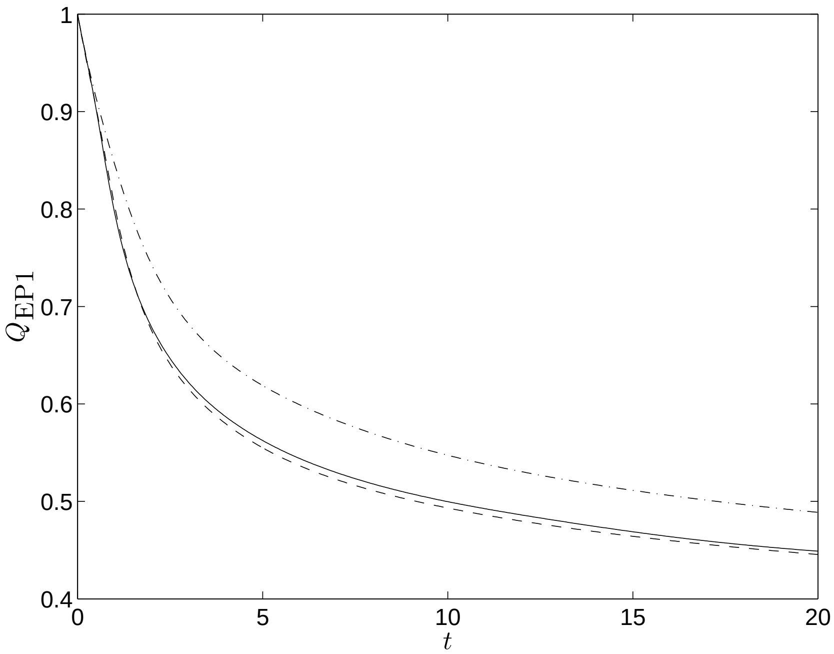


Figure 7.3: CSI1, CSI2, CSI3; EP1: Conservation of kinetic energy. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

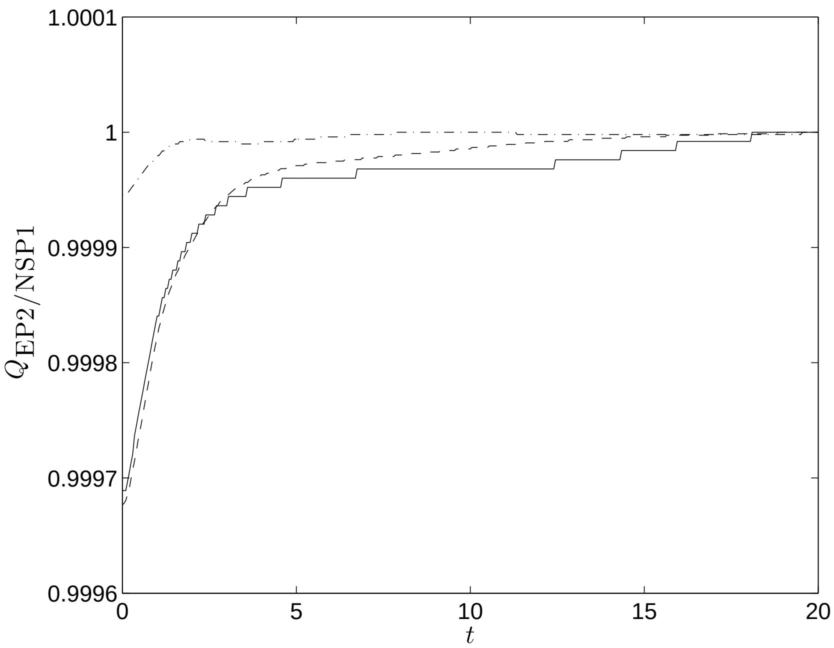


Figure 7.4: CSI1, CSI2, CSI3; EP2/NSP1: Conservation of the z-projection of momentum. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

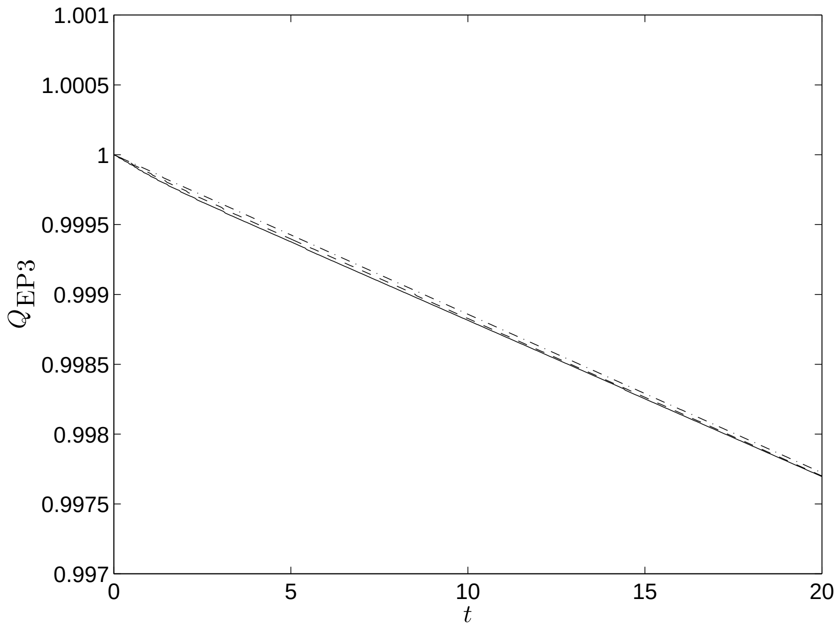


Figure 7.5: CSI1, CSI2, CSI3; EP3: Conservation of the z-projection of the angular momentum. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

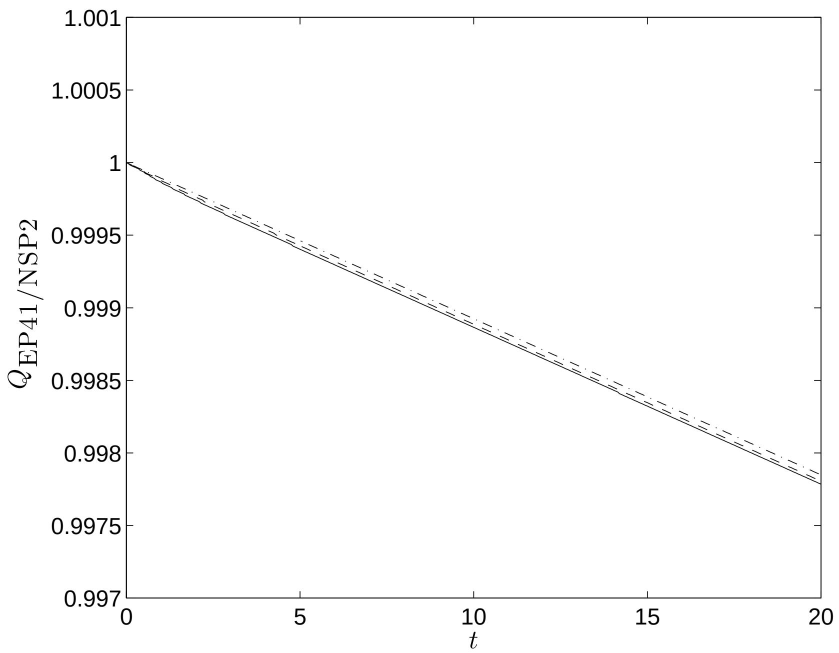


Figure 7.6: CSI1, CSI2, CSI3; EP41/NSP2: Conservation of the generalized momenta/angular momenta, $n = 1$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

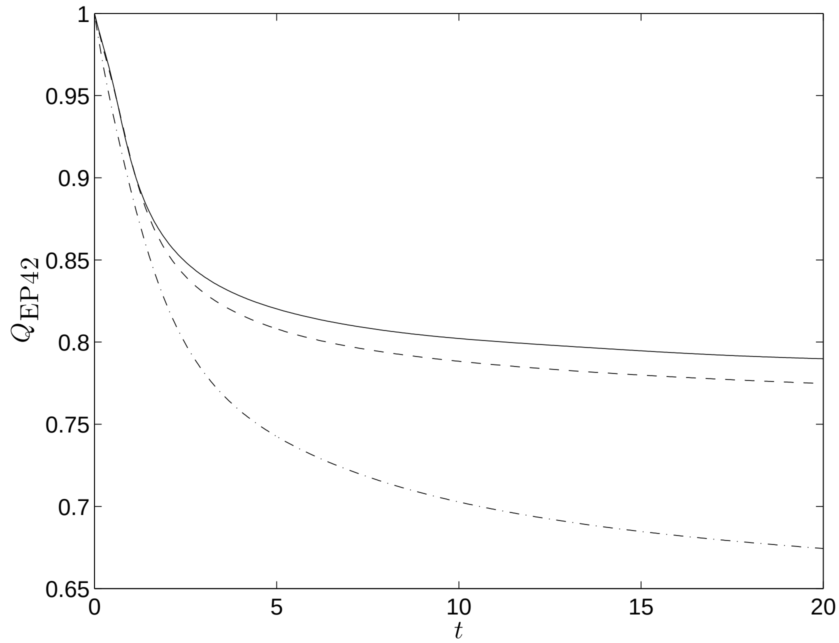


Figure 7.7: CSI1, CSI2, CSI3; EP42: Conservation of the generalized momenta/angular momenta, $n = 2$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

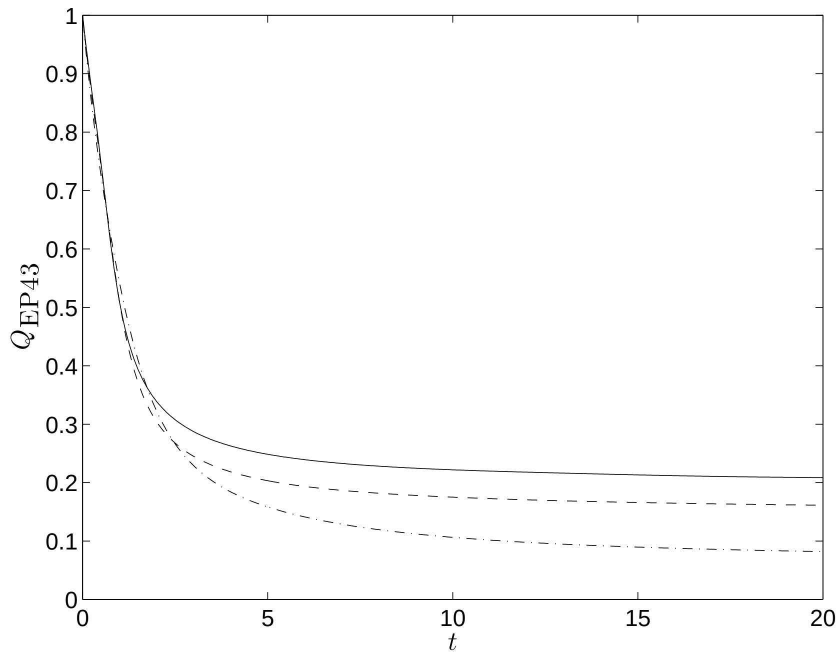


Figure 7.8: CSI1, CSI2, CSI3; EP43: Conservation of the generalized momenta/angular momenta, $n = 3$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

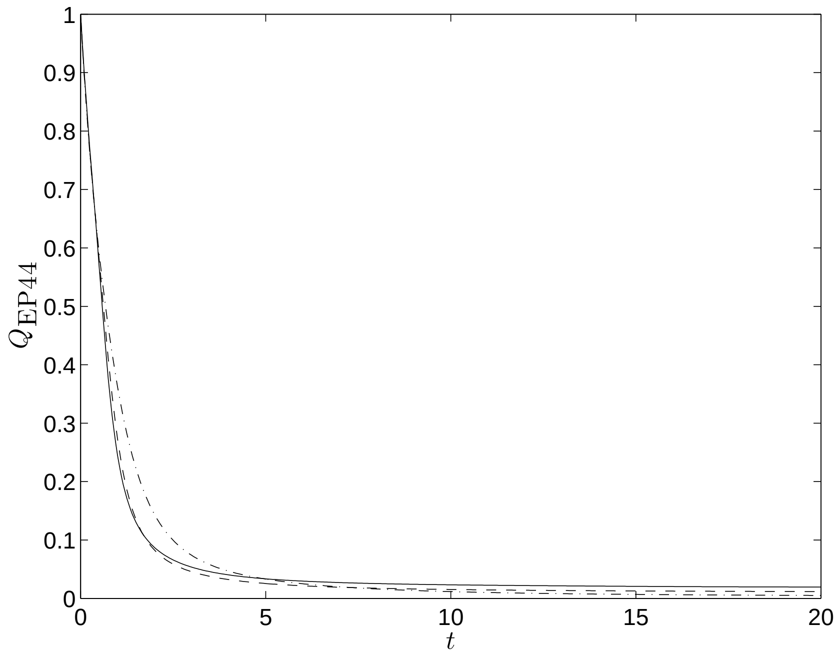


Figure 7.9: CSI1, CSI2, CSI3; EP44: Conservation of the generalized momenta/angular momenta, $n = 4$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

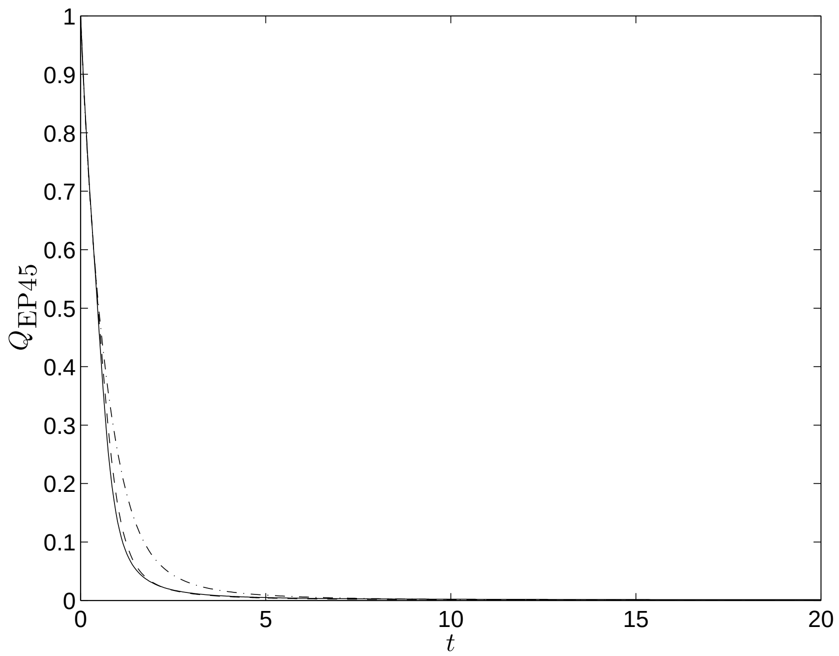


Figure 7.10: CSI1, CSI2, CSI3; EP45: Conservation of the generalized momenta/angular momenta, $n = 5$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

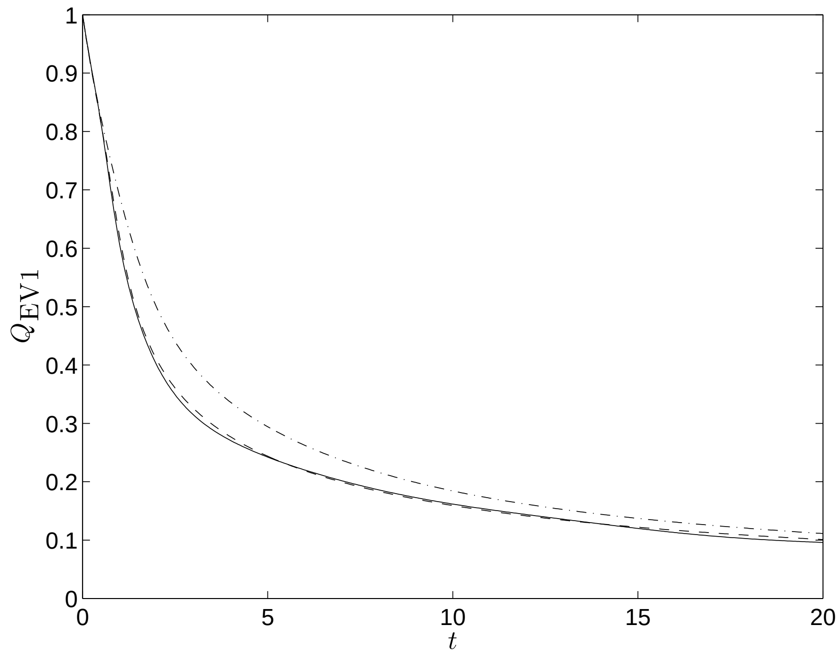


Figure 7.11: CSI1, CSI2, CSI3; EV1: Conservation of helicity. Solid line corresponds to $X_L = 0.5$, dashed line shows the case the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

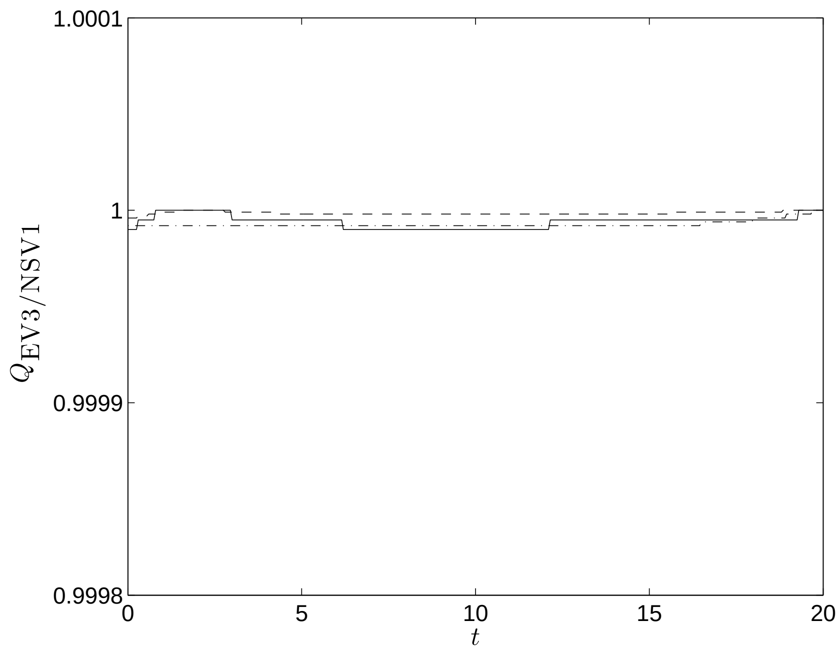


Figure 7.12: CSI1, CSI2, CSI3; EV3/NSV1: A family of vorticity conservation laws involving ω^{φ} plotted for $Q(t) = 1$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

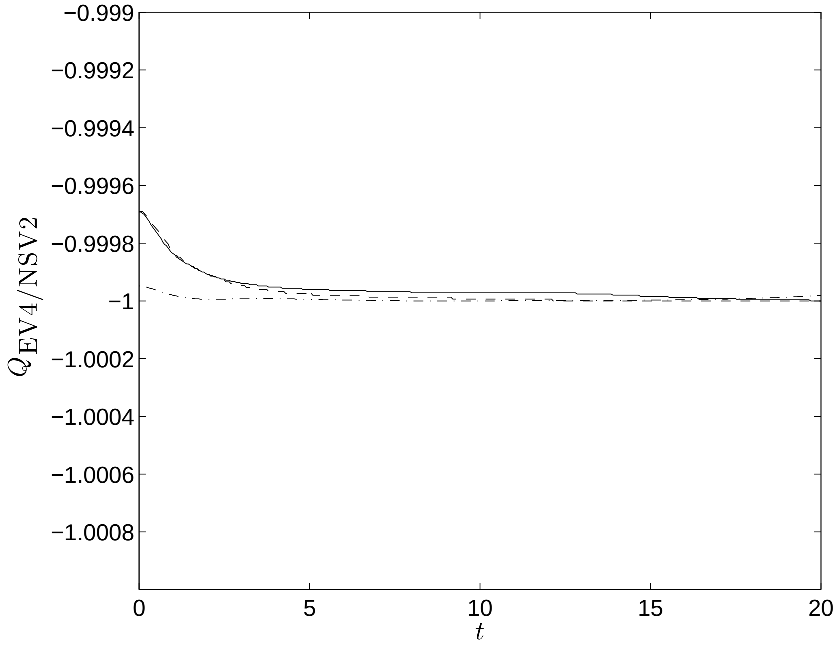


Figure 7.13: CSI1, CSI2, CSI3; EV4/NSV2: Vorticity conservation law. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

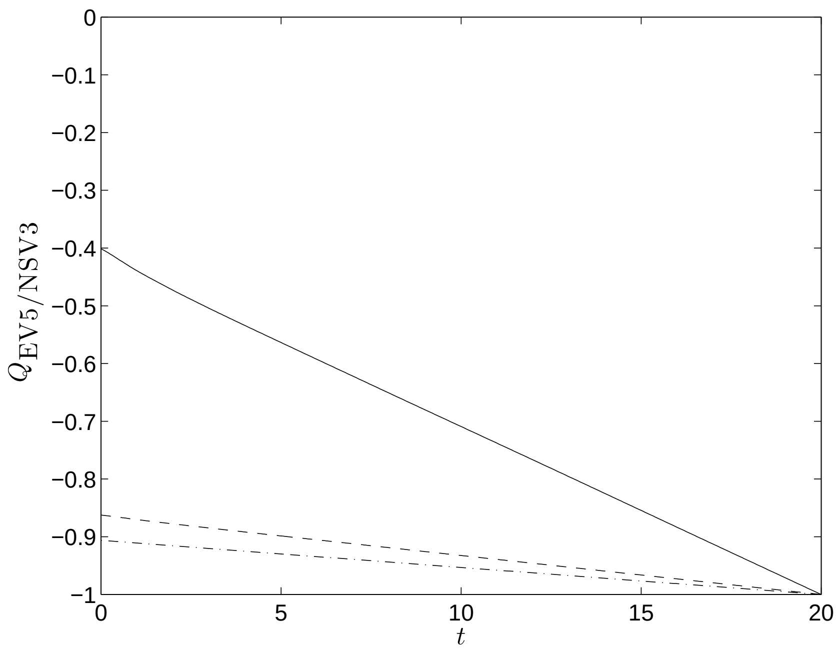


Figure 7.14: CSI1, CSI2, CSI3; EV5/NSV3: Vorticity conservation law. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

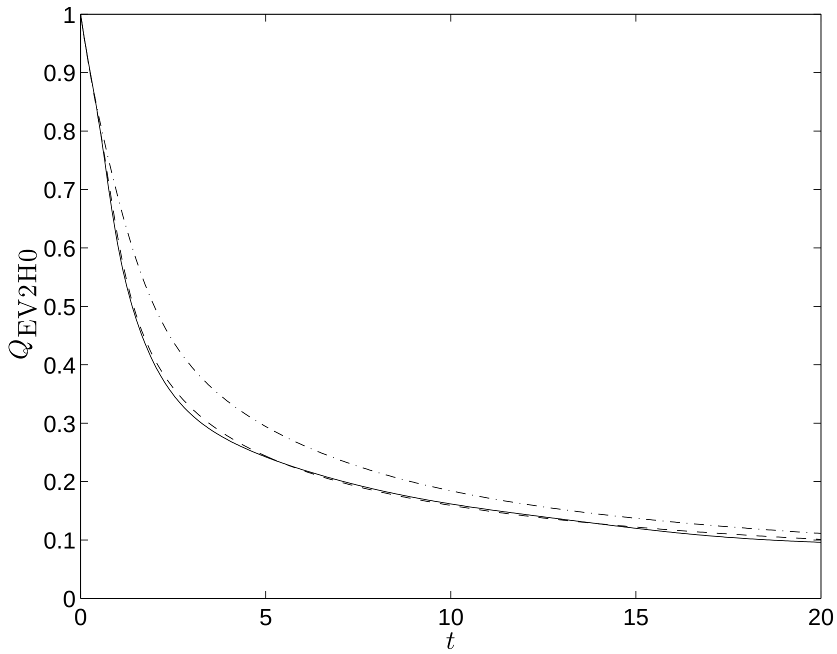


Figure 7.15: CSI1, CSI2, CSI3; EV2H0: An infinite family of generalized helicity conservation laws, $n = 0$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

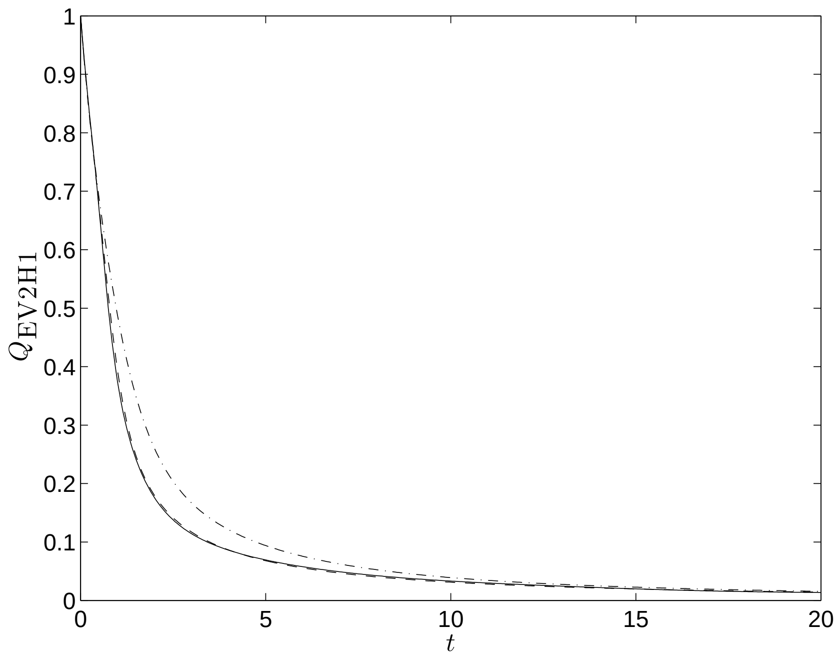


Figure 7.16: CSI1, CSI2, CSI3; EV2H1: An infinite family of generalized helicity conservation laws, $n = 1$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

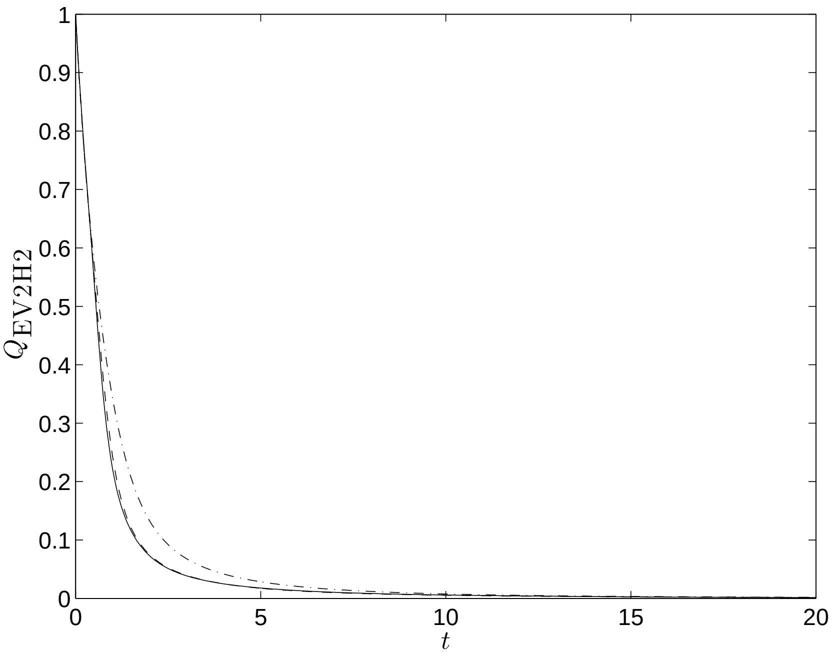


Figure 7.17: CSI1, CSI2, CSI3; EV2H2: An infinite family of generalized helicity conservation laws, $n = 2$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

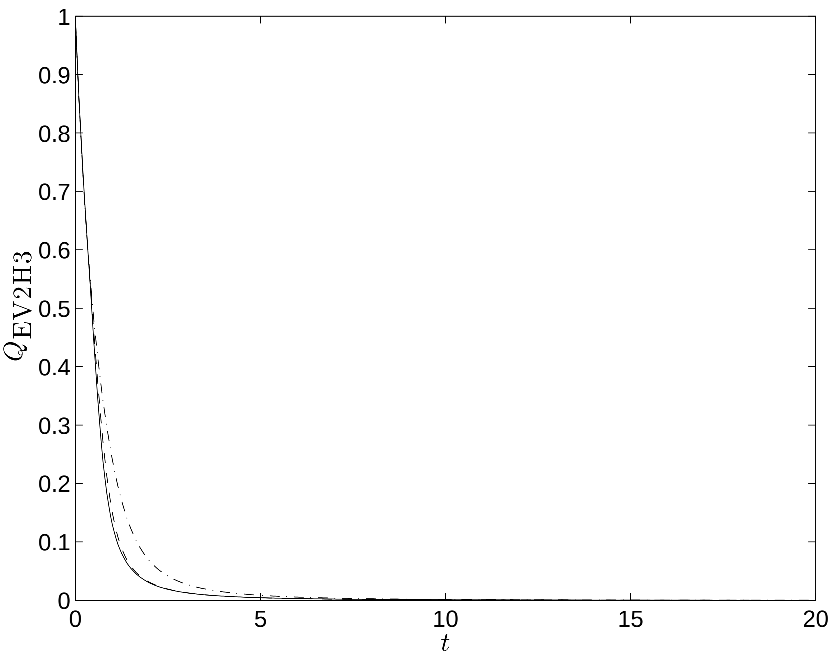


Figure 7.18: CSI1, CSI2, CSI3; EV2H3: An infinite family of generalized helicity conservation laws, $n = 3$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

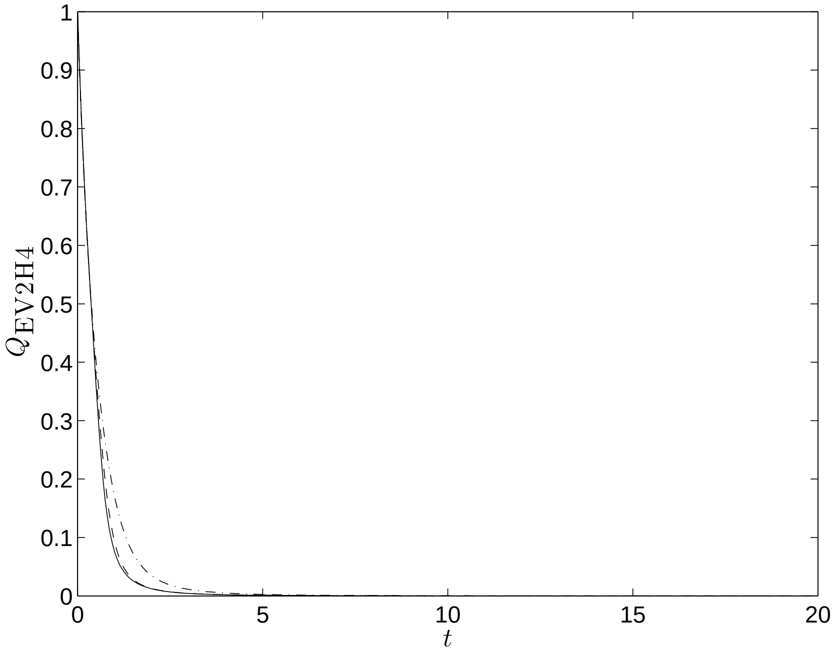


Figure 7.19: CSI1, CSI2, CSI3; EV2H4: An infinite family of generalized helicity conservation laws, $n = 4$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

7.3.2 Random IC

In case of random initial condition only the simulations for $Re = 1000$ are presented. Increasing the Reynolds number to $Re = 5000$ leads to heavy oscillations and non-physical behaviour. The parameters, which are equal for all simulations presented in this subsection, are the following:

$$N_r = N_{th} = 372, \quad \Delta r = 0.01612, \quad \Delta t = 0.0005, \quad N_{It} = 10000,$$

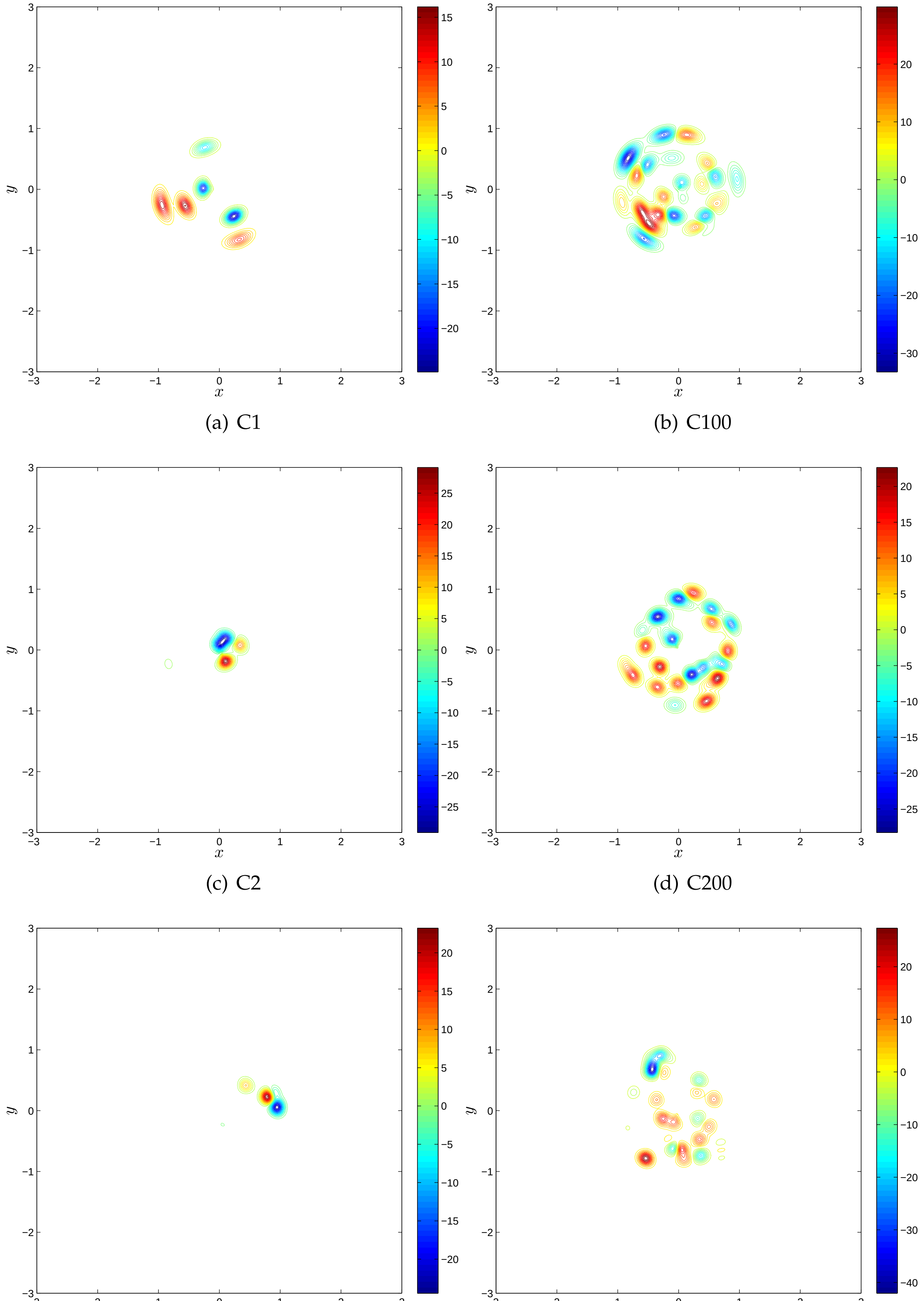
$$Re = 1000, \quad R_{ext} = 6.0, \quad R_0 = 1.0, \quad A_0 = 0.1.$$

The reduced pitch X_L and the number of vortices N_{vort} were varied as follows:

$$\begin{array}{llllll} \text{C1}: & X_L = 0.5, & N_{vort} = 6; & \text{C100}: & X_L = 0.5, & N_{vort} = 30; \\ \text{C2}: & X_L = 1.0, & N_{vort} = 6; & \text{C200}: & X_L = 1.0, & N_{vort} = 30; \\ \text{C3}: & X_L = 2.0, & N_{vort} = 6; & \text{C300}: & X_L = 2.0, & N_{vort} = 30. \end{array}$$

For the numerical results in case of random initial condition the following conclusions can be made: for the simulations C1, C2, C3 the results from the previous subsections can be transferred, the integral quantities of all simulated conservation laws become constant in time not later than thirty percent of the computational time except the conservation of vorticity related quantities EV3/NSV1, EV4/NSV2 and EV5/NSV3, which is not satisfied anymore. Furthermore, the influence of the helical pitch disap-

pears; the results for three different pitches are similarly satisfying. The simulations C100, C200, C300 reproduce the results in a similar manner, in addition, the conservation of EV3/NSV1, EV4/NSV2 and EV5/NSV3 seems to hold, even though an oscillation at the beginning of the simulations occurs.
As already mentioned, increasing the Reynolds number from $Re = 1000$ to $Re = 5000$ leads to not acceptable oscillations for the conservation of EV3/NSV1, EV4/NSV2, EV5/NSV3; the remaining quantities shows oscillations as well, whether of smaller magnitude. The randomness of the initial condition could be one possible reason for the oscillations. Also the order of the numerical method (second order) is a supposable explanation.



(a) C1 (b) C100

(c) C2 (d) C200

(e) C3 (f) C300

Figure 7.20: Contour plots of the vorticity component ω^η for different simulations at $t = 0$, $z = 0$.

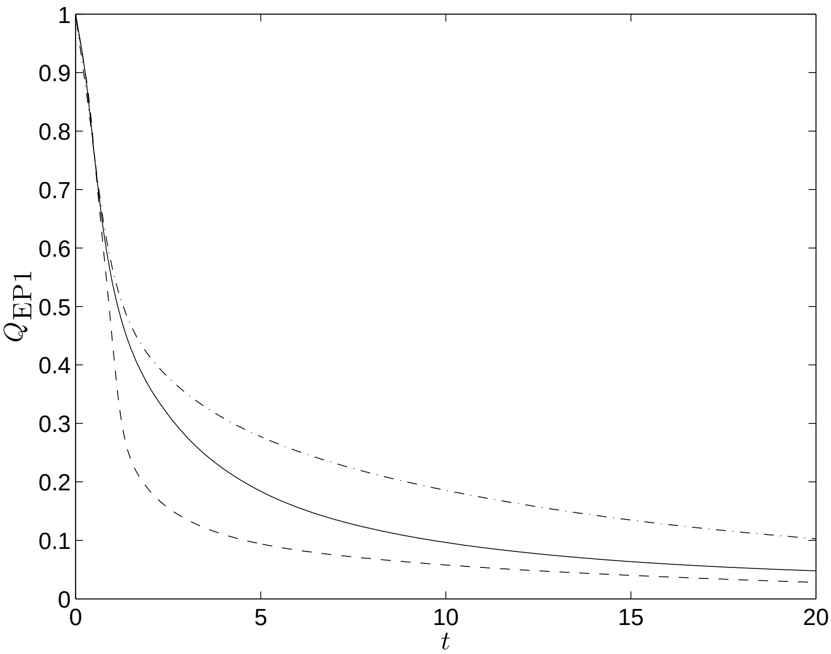


Figure 7.21: C1, C2, C3; EP1: Conservation of kinetic energy. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

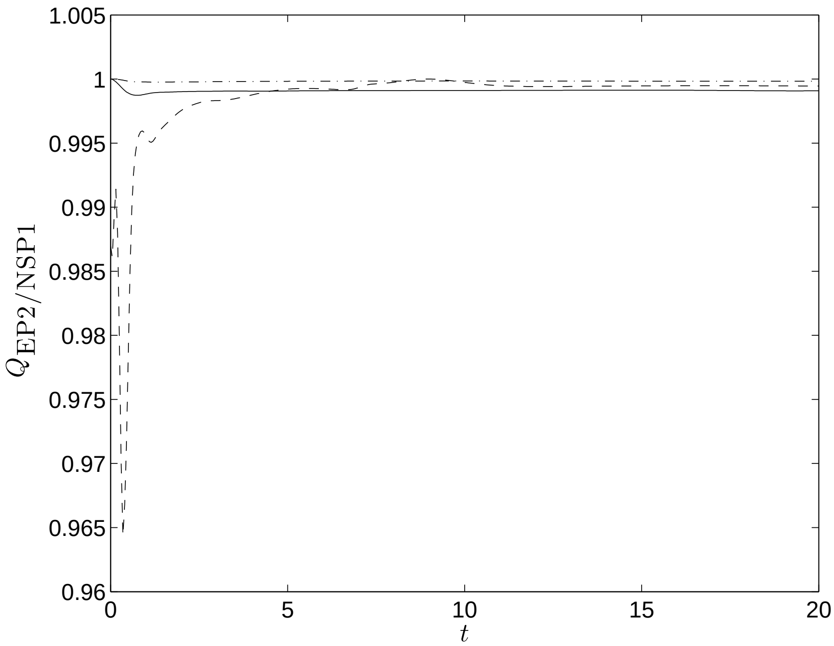


Figure 7.22: C1, C2, C3; EP2/NSP1: Conservation of the z-projection of momentum. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

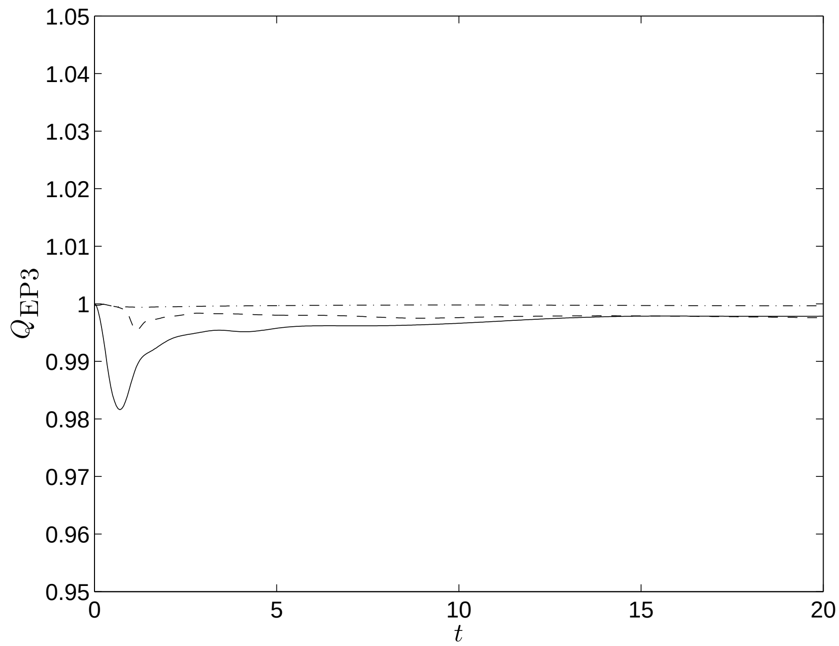


Figure 7.23: C1, C2, C3; EP3: Conservation of the z-projection of the angular momentum. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

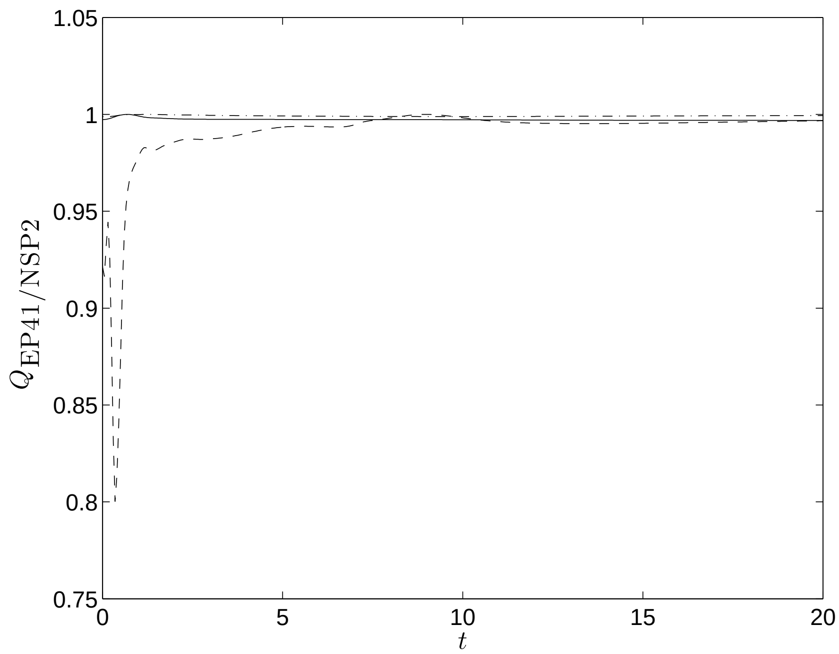


Figure 7.24: C1, C2, C3; EP41/NSP2: Conservation of the generalized momenta/angular momenta, $n = 1$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

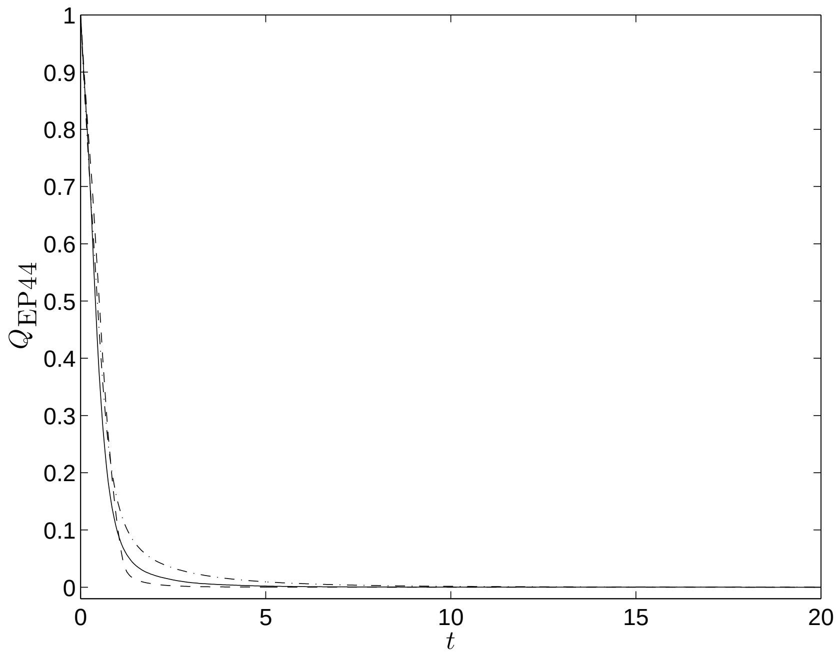


Figure 7.25: C1, C2, C3; EP44: Conservation of the generalized momenta/angular momenta, $n = 4$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

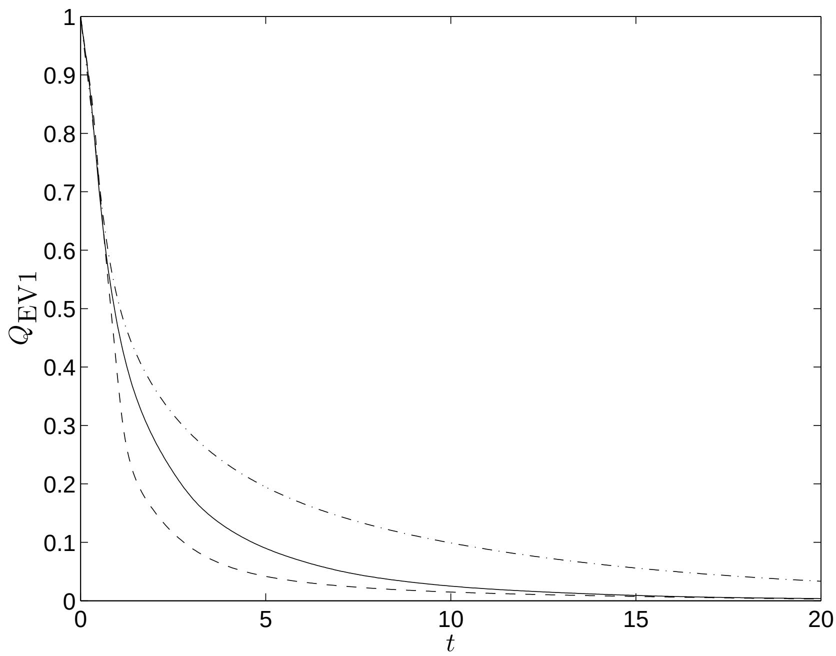


Figure 7.26: C1, C2, C3; EV1: Conservation of helicity. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

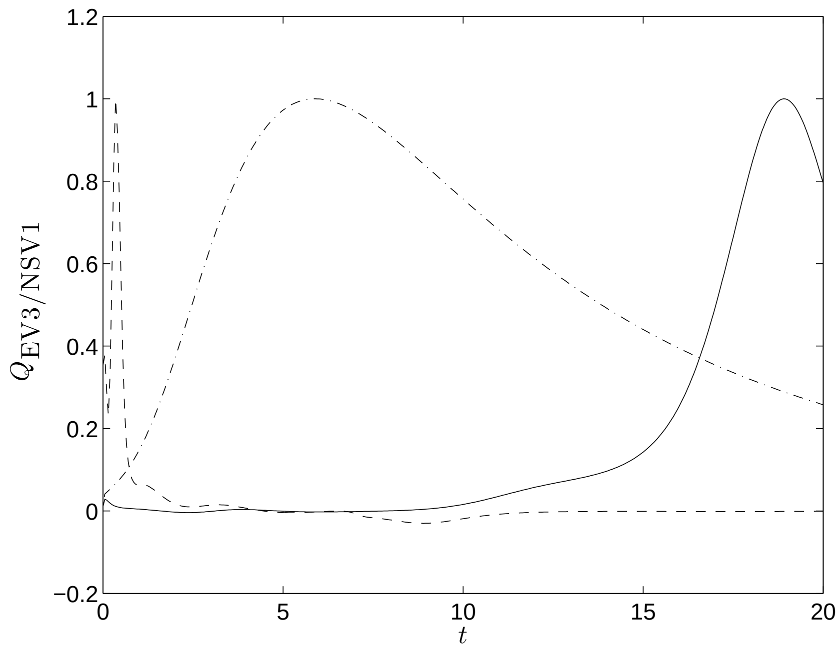


Figure 7.27: C1, C2, C3; EV3/NSV1: A family of vorticity conservation laws involving ω^{φ} plotted for $Q(t) = 1$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

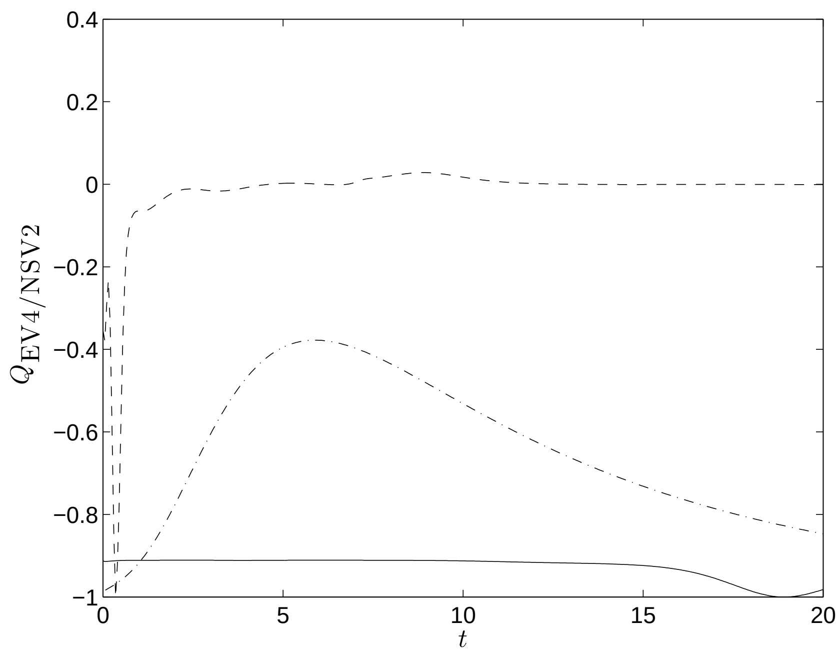


Figure 7.28: C1, C2, C3; EV4/NSV2: Vorticity conservation law. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

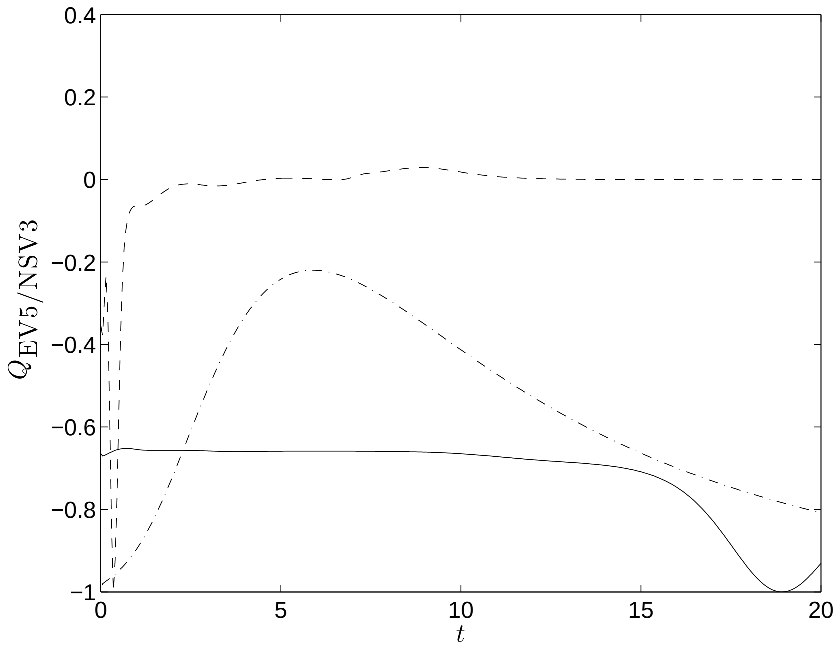


Figure 7.29: C1, C2, C3; EV5/NSV3: Vorticity conservation law. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

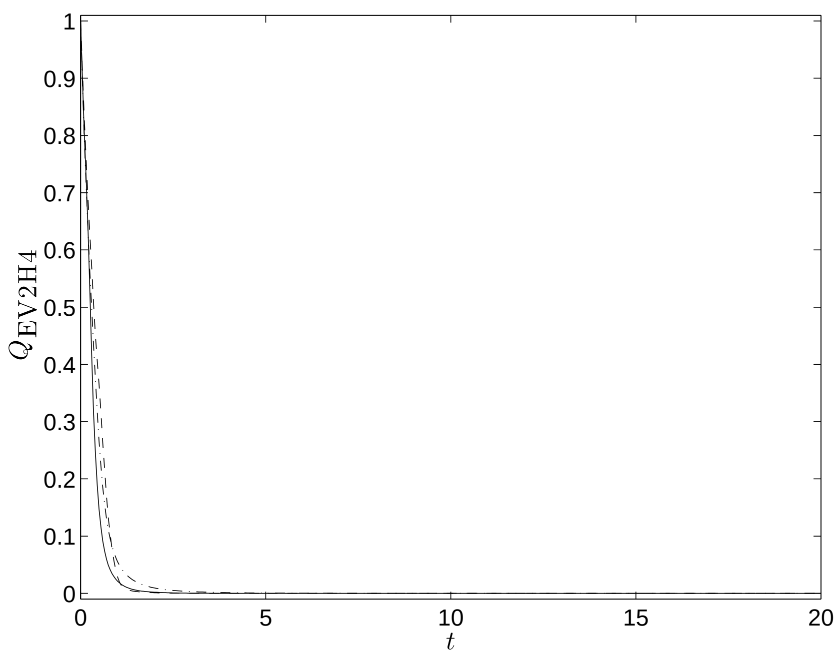


Figure 7.30: C1, C2, C3; EV2H4: An infinite family of generalized helicity conservation laws, $n = 4$. Solid line corresponds to the case $X_L = 0.5$, dashed line shows the case $X_L = 1$, dashed-dotted line represents the case $X_L = 2$.

8 Summary and conclusions

Incompressible helically symmetric flows that play an important role in various natural, applied and laboratory settings have been considered in the present dissertation. In cylindrical coordinates (r, φ, z), a helical variable is given by $\xi = az + b\varphi$; with curves $\xi = const.$ describing helices. In a helically invariant setting, all physical quantities are restricted depend only on time t, the cylindrical radius r and the helical variable ξ.

In the current contribution, the full set of helically invariant Navier-Stokes equations was derived both in primitive variables (formulae (2.13)) and in the vorticity formulation (formulae (2.33)). Important special cases of rotational and plane symmetry arise in the limiting cases of helical parameters $a = 1,\ b = 0$ and $a = 0,\ b = 1$, respectively. The corresponding reductions of the Navier-Stokes equations were derived in section 2.2.1 and 2.2.2.

In general, helically symmetric, rotationally symmetric, and plane symmetric flows have all three velocity components nonzero, and hence are often called "$2\frac{1}{2}$-dimensional flows". Many applications use two-component flows, where the velocity component in the invariant direction vanishes. Such flows were also considered in the current dissertation.

The direct construction method was applied to systematically seek local conserved quantities and the corresponding fluxes of conservation laws that hold for the models listed above. Well-known conservation laws, such as conservation of momentum, angular momentum, energy and helicity for inviscid flows, were reproduced. In addition, several new families of conservation laws were derived, which are specific to the helically invariant setting, both in the viscous and in the inviscid case, as follows:

1. For helically invariant Euler equations in primitive variables (section 4.1), conservation laws of kinetic energy and $z-$projections of momentum and angular momentum hold (formulae (4.1), (4.2) and (4.3)). In addition, a new infinite family of generalized momentum/angular momentum conservation laws (4.4) was discovered. All conservation laws in this family are material conservation laws (1.3), corresponding to the conservation of the quantity $F\left(\frac{r}{B}u^\eta\right)$ initially assigned to any moving fluid parcel, for an arbitrary function $F(\cdot)$.

2. For helically invariant Euler equations in the vorticity formulation (section 4.2), the conservation of helicity h (4.6) is readily obtained. In the current contribution, a new family of generalized helicity conservation laws (4.9) was derived, with the conserved quantity given by $h\,H\left(\frac{r}{B}u^\eta\right)$ for an arbitrary function $H(\cdot)$.

These non-material conservation laws have not been observed before in any setting.

Moreover, a new infinite family of vorticity-related conserved quantities (4.10) was found, as well as three additional conservation laws given by (4.11), (4.13) and (4.14), involving combinations of vorticity components and spatial variables. These conservation laws hold in the inviscid case, as well as in the viscous case after an appropriate extension.

3. Conserved quantities for the helically invariant viscous flows were considered in section 5. Remarkably, a $z-$projection of momentum, and an additional momentum-like quantity $(r/B)u^{\eta}$ are preserved even by a viscous flow (formulae (5.1) and (5.2)).

4. In the vorticity formulation (section 5.2), the helically invariant viscous flow equations were found to possess a remarkable set of vorticity-related conservation laws, including the family (5.3) and single conservation laws (5.4) and (5.5), that directly generalize the corresponding inviscid ones onto the case $\nu > 0$.

An important restriction that is often considered in literature in various settings is the case of two-component flows, where one of the velocity components vanishes identically. In chapter 6 of the present dissertation, one considered two-component helically invariant flows, with the velocity component in the invariant direction $u^{\eta} \equiv 0$. With such a restriction, the governing equations in primitive variables (2.13) and in the vorticity formulation (2.33) significantly simplify; in particular, two vorticity components vanish identically: $\omega^r = \omega^{\xi} \equiv 0$. Conservation laws for this setting were computed as follows.

5. Due to the differential constraint (6.2c), only inviscid flows have been considered in the general helically invariant setting with $a, b \neq 0$. For such flows, an infinite set of enstrophy-related vorticity conservation laws (6.6) was discovered. For the plane flow equations, the family reduces to (6.16), which is already known in literature. However, its full helical form (6.6) and the axially symmetric reduction (6.22) first appear in the new results of the current contribution.

6. For classical two-component plane flows (section 6.2), both in viscous and inviscid settings, one obtains additional conservation laws that do not hold for a general helical setting. In particular, in primitive variables, one has the conservation of angular momentum in the $z-$ direction given by (6.9), and two families of conservation laws (6.10) and (6.11) corresponding to the "center of mass theorem" and involving arbitrary functions of time. The latter families have been previously known to hold only in the inviscid setting. In the vorticity formulation, one additionally has a conservation law (6.12), and three families of conservation laws (6.13), (6.14) and (6.15), involving arbitrary functions of time. The latter three families have been previously known to hold only for special values of the arbitrary functions, and only in the inviscid setting.

7. For axisymmetric two-component flows (section 6.3), in both viscous and inviscid settings, three new families of conservation laws were derived: equation

(6.19) in primitive variables and equation (6.20) and (6.21) in the vorticity formulation.

With the DNS code HELIX the results of the theoretical investigations concerning local conservation laws of helically symmetric flows could be integrated to see if the conservation is a global property. For the results of the simulations it can be deduced, that the most of the local conserved quantities become global. The best results could be achieved for the simulations with $Re = 1000$ and regular initial conditions. The conservation property for the quantities, which contain the vorticity, become worse with increasing Reynolds number, the use of random initial condition impaired these results further. The variation of the reduced helical pitch did not seem to have strong influence to the time evolution of the integrals of conserved quantities. The oscillations, which occur in the simulations with random initial condition and become stronger with higher Reynolds number, could be presumably avoided by changing the initial conditions, e.g. instead of a large number of small vortices a small number of vortices with big vortex core size could be used. Another possibility to perform simulations for higher Reynolds number could be achieved by a numerical scheme of higher order.

In summary, the assumption of helical invariance gives rise to additional infinite new families of conservation laws of fluid flow equations, in a variety of settings, including cases with nonzero viscosity. Many of the new conservation laws are vorticity related. (No additional conservation laws were found using the stream function formulation.) These results make the helical invariance property seem to be a particularly important ansatz for solution of fluid dynamics equations. Based on the findings presented in the current dissertation, one may argue that the fact that helically invariant flows occur frequently in observed phenomena may be related to their special structure, which mathematically reveals itself through additional infinite families of conserved quantities.

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A Appendix

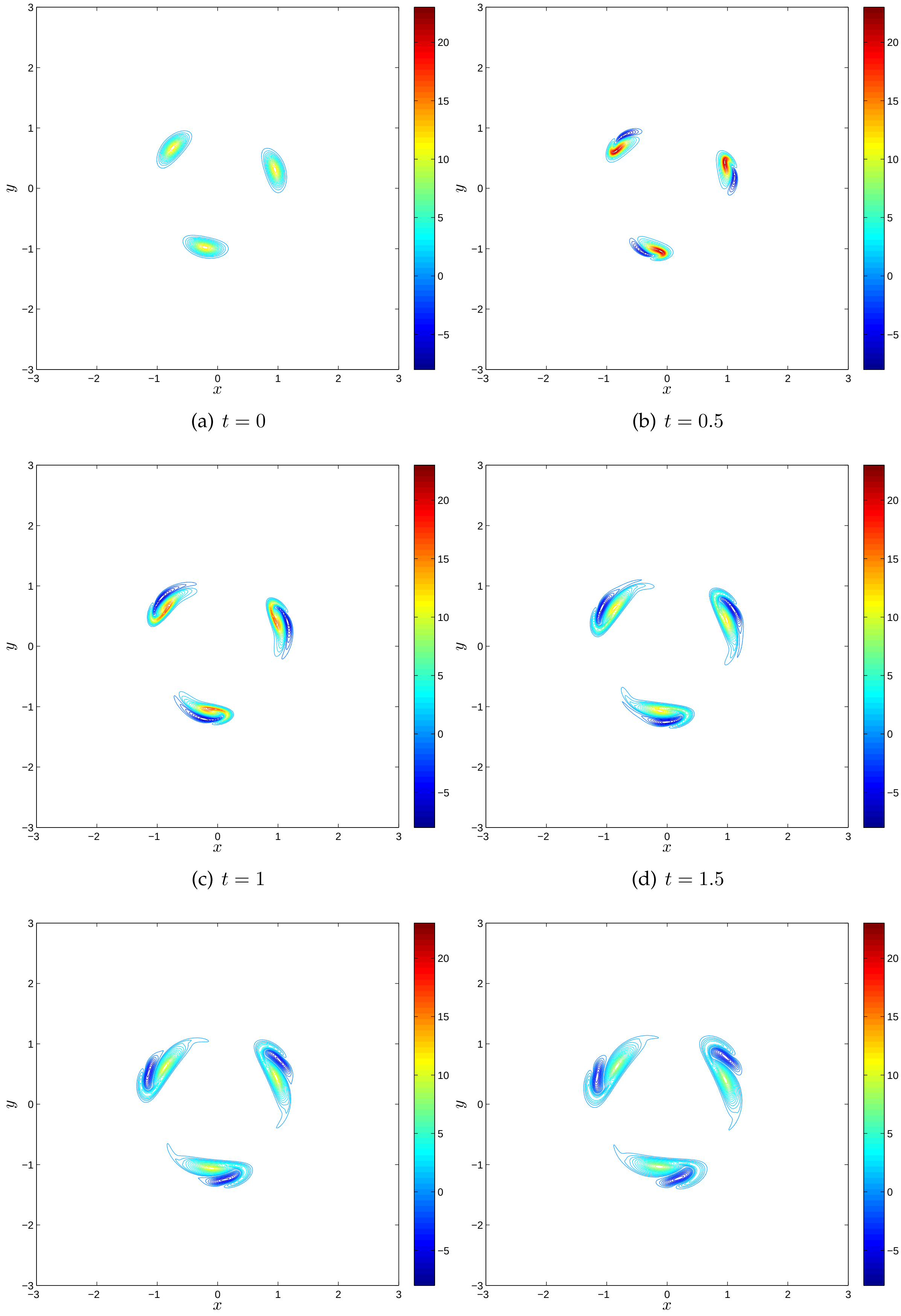
The appendix is structured in two parts. First, in Table A.1 and A.2 one can find the order of magnitude of the flux across the boundary at $r = 5.5$ for all simulations, which were presented whithin this thesis. Second, the iso contour plots of the vorticity component ω^η in a plane at $z = 0$ are shown here to get an information about the time evolution.

Flux	CSI1	CSI2	CSI3	CSI500	CSI600	CSI700
EP2	10^{-30}	10^{-26}	10^{-23}	10^{-25}	10^{-33}	10^{-34}
EP3	10^{-27}	10^{-24}	10^{-22}	10^{-20}	10^{-28}	10^{-31}
EP41	10^{-27}	10^{-24}	10^{-21}	10^{-20}	10^{-29}	10^{-31}
EP42	10^{-28}	10^{-24}	10^{-23}	10^{-19}	10^{-29}	10^{-33}
EP43	10^{-29}	10^{-25}	10^{-23}	10^{-19}	10^{-29}	10^{-31}
EP44	10^{-30}	10^{-27}	10^{-24}	10^{-20}	10^{-29}	10^{-31}
EP45	10^{-30}	10^{-27}	10^{-25}	10^{-20}	10^{-29}	10^{-32}
EV3	10^{-31}	10^{-27}	10^{-24}	10^{-22}	10^{-39}	10^{-42}
EV4	10^{-29}	10^{-25}	10^{-23}	10^{-20}	10^{-33}	10^{-34}
EV5	10^{-27}	10^{-24}	10^{-21}	10^{-18}	10^{-28}	10^{-30}
NSP1	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-7}
NSP2	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
NSV1	10^{-17}	10^{-16}	10^{-16}	10^{-15}	10^{-17}	10^{-13}
NSV2	10^{-15}	10^{-16}	10^{-16}	10^{-13}	10^{-14}	10^{-13}
NSV3	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}

Table A.1: Order of magnitude of the flux across the boundary.

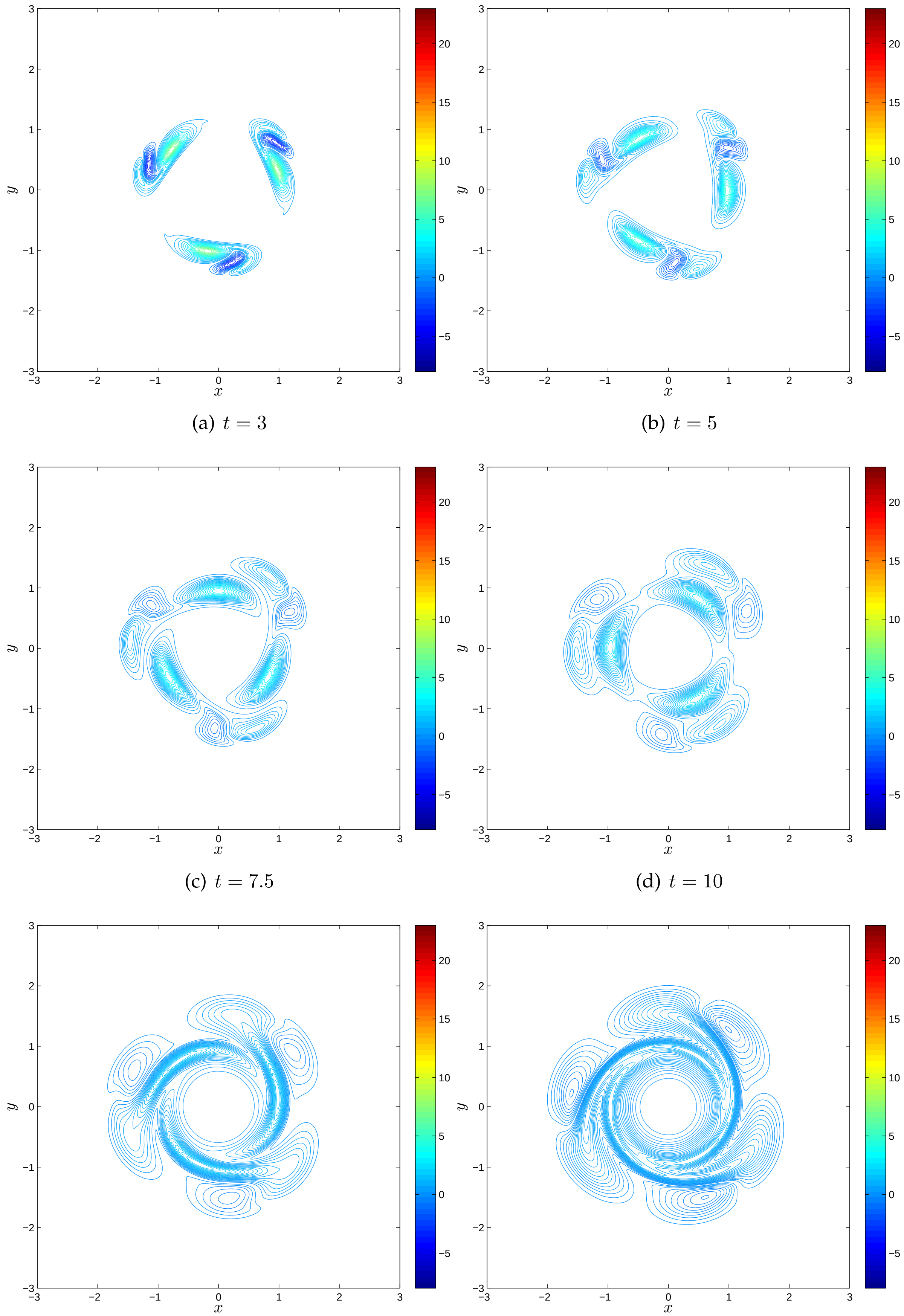
Flux	C1	C2	C3	C100	C200	C300
EP2	10^{-25}	10^{-22}	10^{-20}	10^{-24}	10^{-20}	10^{-18}
EP3	10^{-25}	10^{-22}	10^{-20}	10^{-24}	10^{-20}	10^{-18}
EP41	10^{-30}	10^{-25}	10^{-26}	10^{-29}	10^{-25}	10^{-25}
EP42	10^{-50}	10^{-45}	10^{-49}	10^{-48}	10^{-47}	10^{-48}
EP43	10^{-80}	10^{-64}	10^{-73}	10^{-68}	10^{-69}	10^{-71}
EP44	10^{-92}	10^{-83}	10^{-96}	10^{-88}	10^{-91}	10^{-93}
EP45	10^{-113}	10^{-103}	10^{-119}	10^{-109}	10^{-113}	10^{-116}
EV3	10^{-27}	10^{-23}	10^{-25}	10^{-25}	10^{-23}	10^{-23}
EV4	10^{-24}	10^{-22}	10^{-20}	10^{-23}	10^{-20}	10^{-18}
EV5	10^{-26}	10^{-22}	10^{-19}	10^{-24}	10^{-20}	10^{-18}
NSP1	10^{-12}	10^{-10}	10^{-13}	10^{-11}	10^{-12}	10^{-13}
NSP2	10^{-7}	10^{-6}	10^{-9}	10^{-6}	10^{-8}	10^{-9}
NSV1	10^{-17}	10^{-18}	10^{-22}	10^{-23}	10^{-19}	10^{-21}
NSV2	10^{-15}	10^{-17}	10^{-20}	10^{-23}	10^{-17}	10^{-18}
NSV3	10^{-7}	10^{-6}	10^{-9}	10^{-6}	10^{-8}	10^{-8}

Table A.2: Order of magnitude of the flux across the boundary.



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 1.5$ (e) $t = 2$ (f) $t = 2.5$

Figure A.1: CSI1: $Re = 1000, X_L = 0.5, N_r = N_{th} = 372, R_{ext} = 6.0$

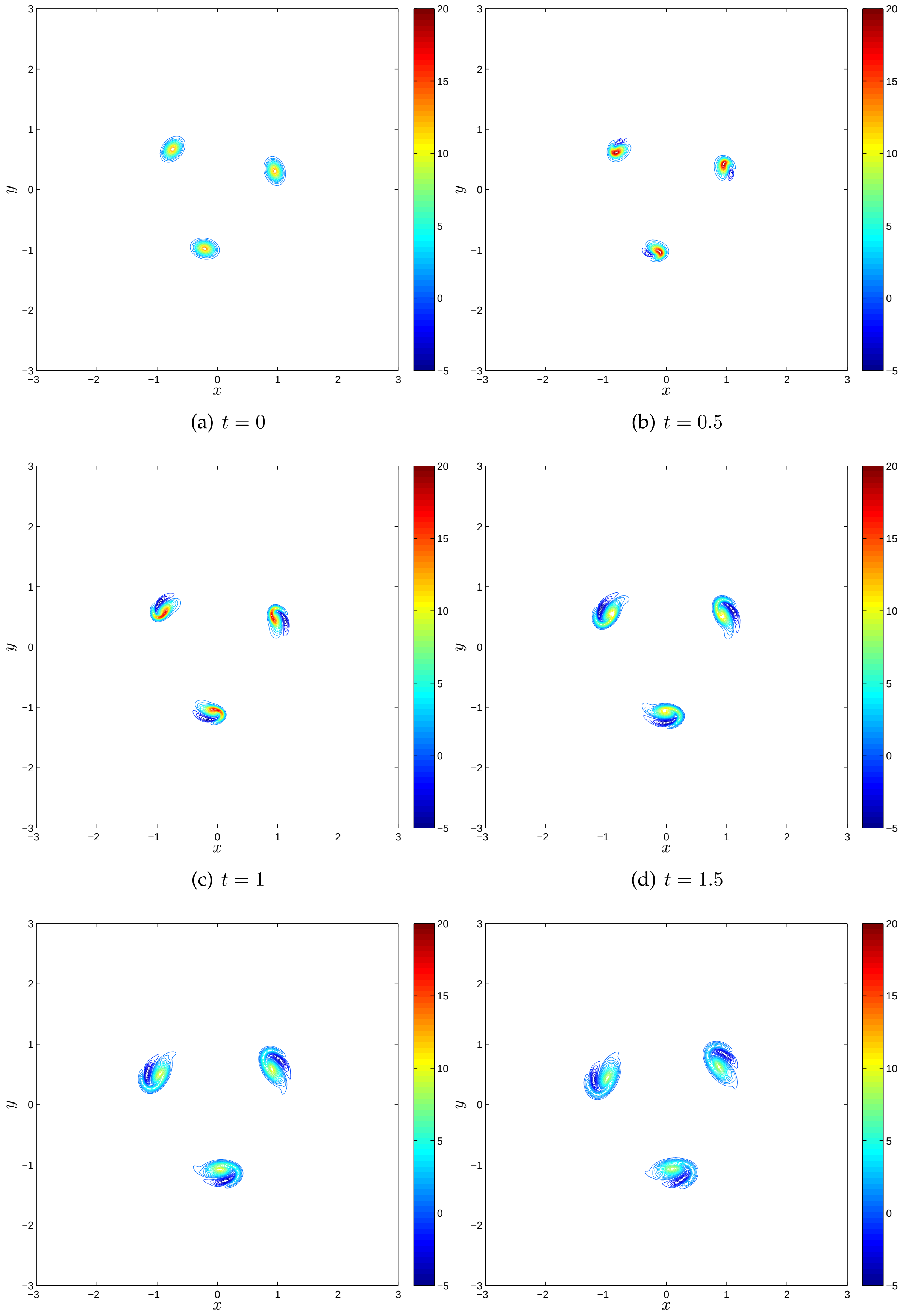


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.2: CSI1: $Re = 1000, X_L = 0.5, N_r = N_{th} = 372, R_{ext} = 6.0$



(a) $t = 0$

(b) $t = 0.5$

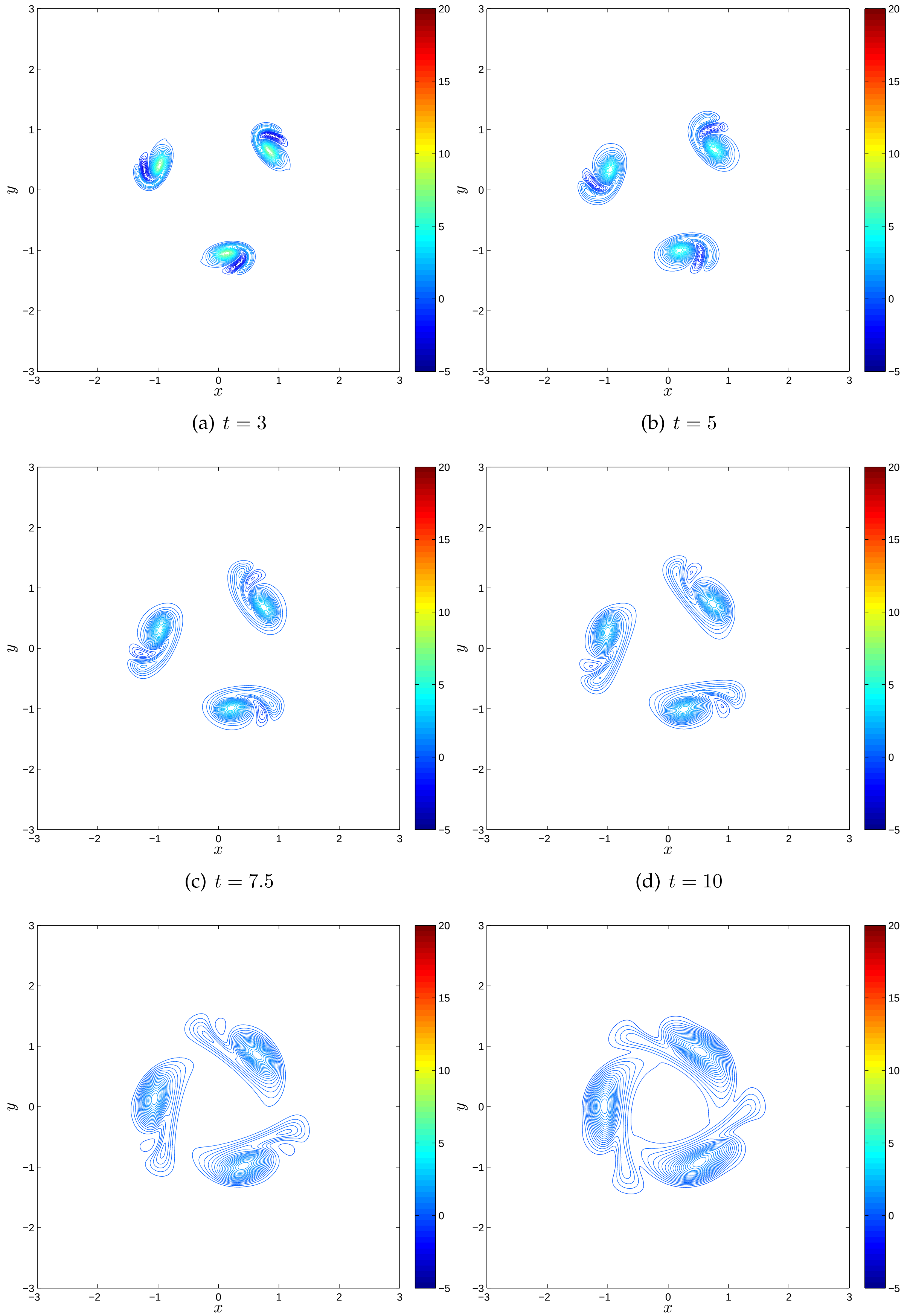
(c) $t = 1$

(d) $t = 1.5$

(e) $t = 2$

(f) $t = 2.5$

Figure A.3: CSI2: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$

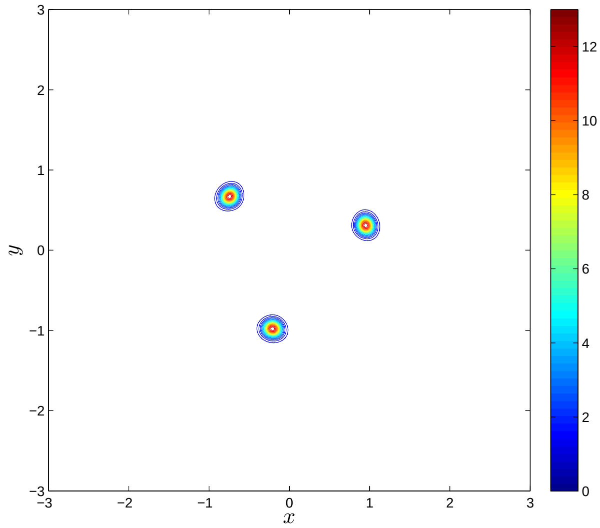


(a) $t = 3$ (b) $t = 5$

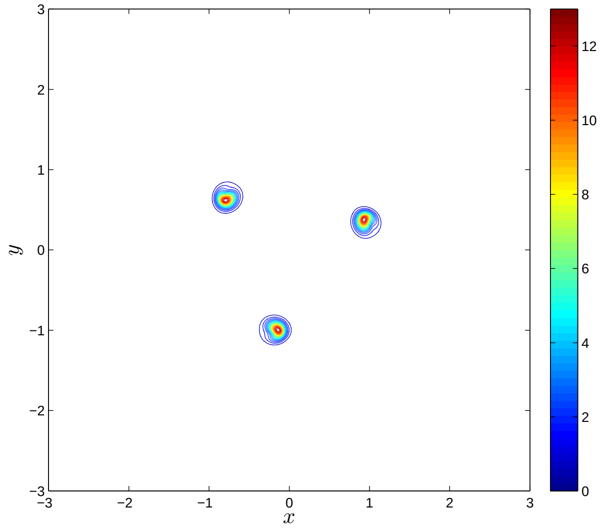
(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

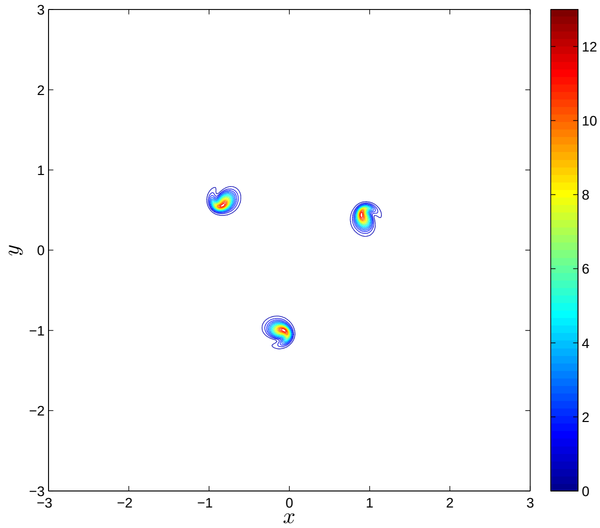
Figure A.4: CSI2: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$



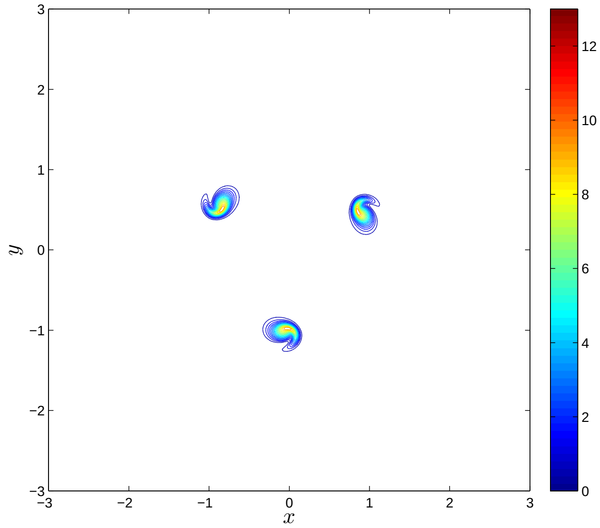
(a) $t = 0$



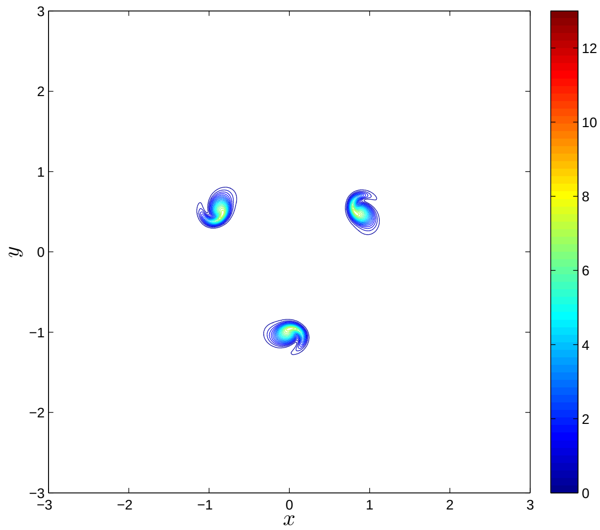
(b) $t = 0.5$



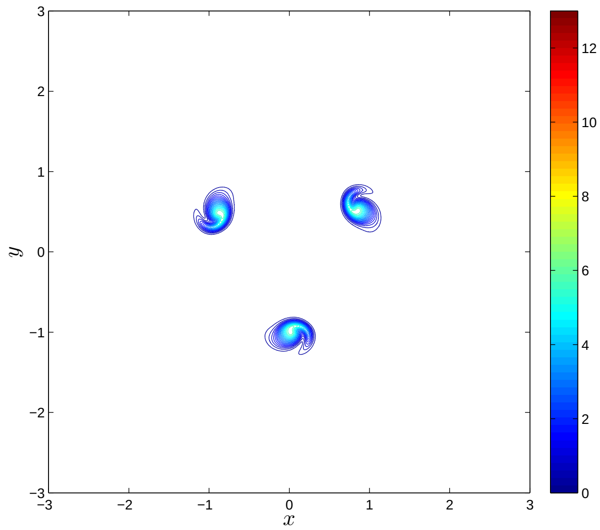
(c) $t = 1$



(d) $t = 1.5$

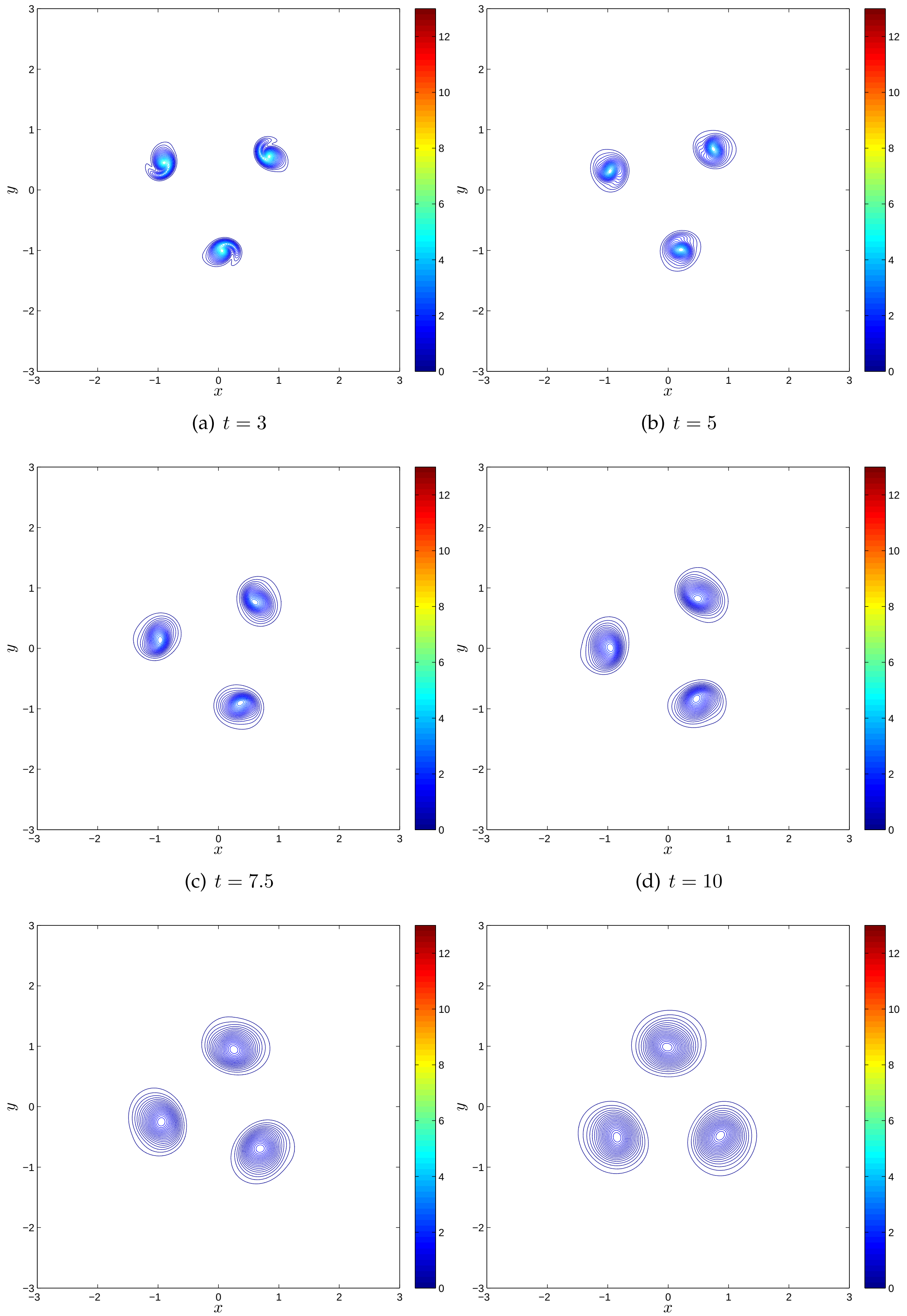


(e) $t = 2$



(f) $t = 2.5$

Figure A.5: CSI3: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$

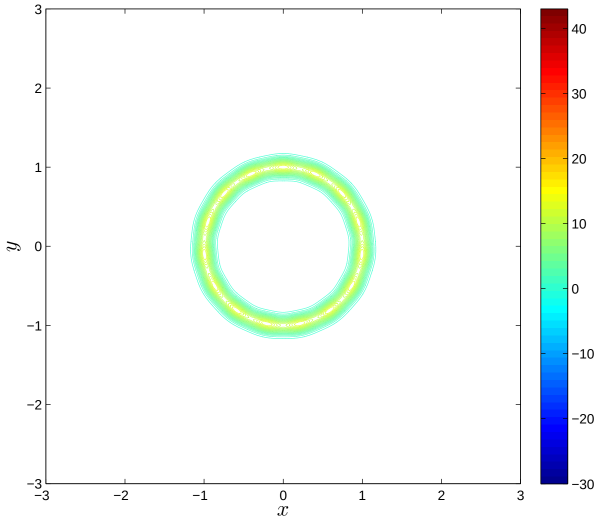


(a) $t = 3$ (b) $t = 5$

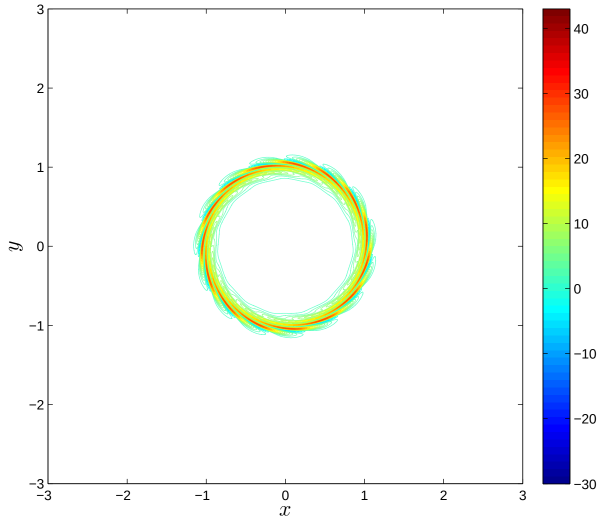
(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

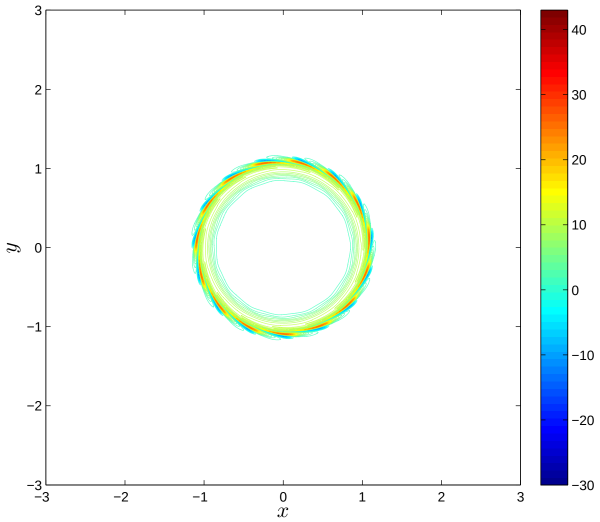
Figure A.6: CSI3: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$



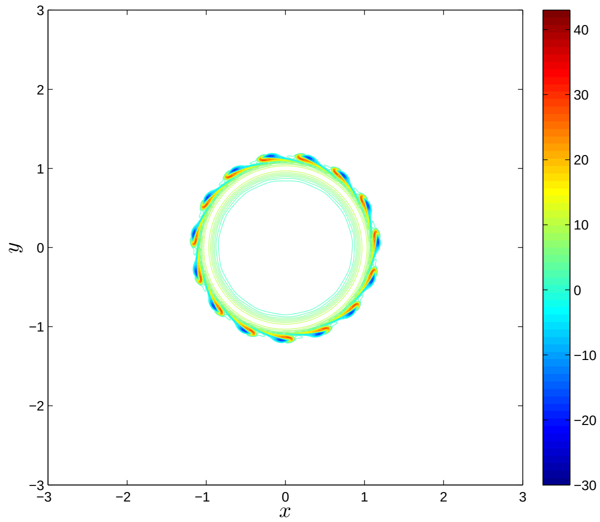
(a) $t = 0$



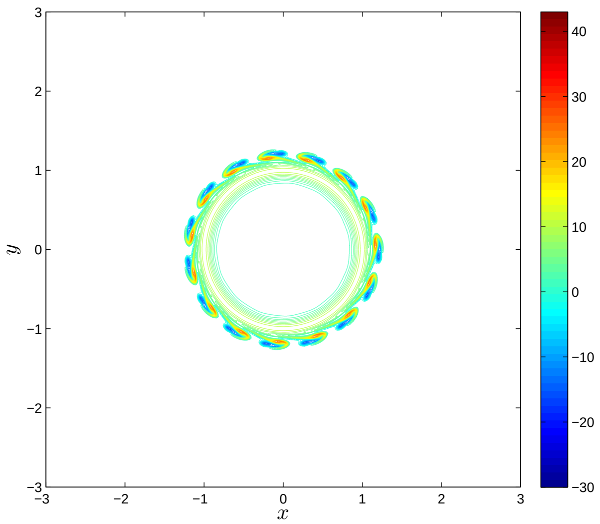
(b) $t = 0.5$



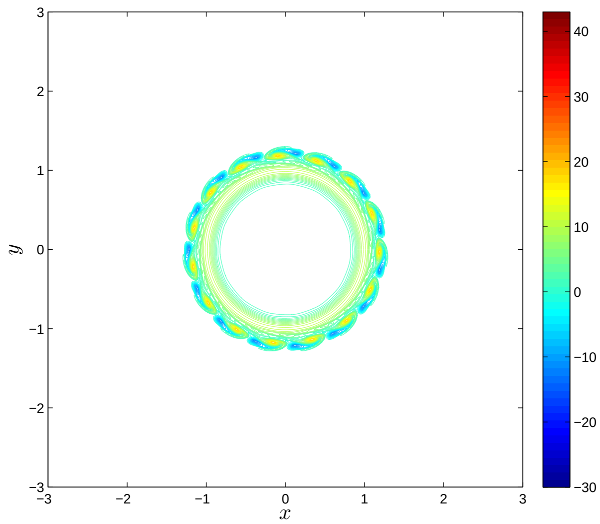
(c) $t = 1$



(d) $t = 1.5$

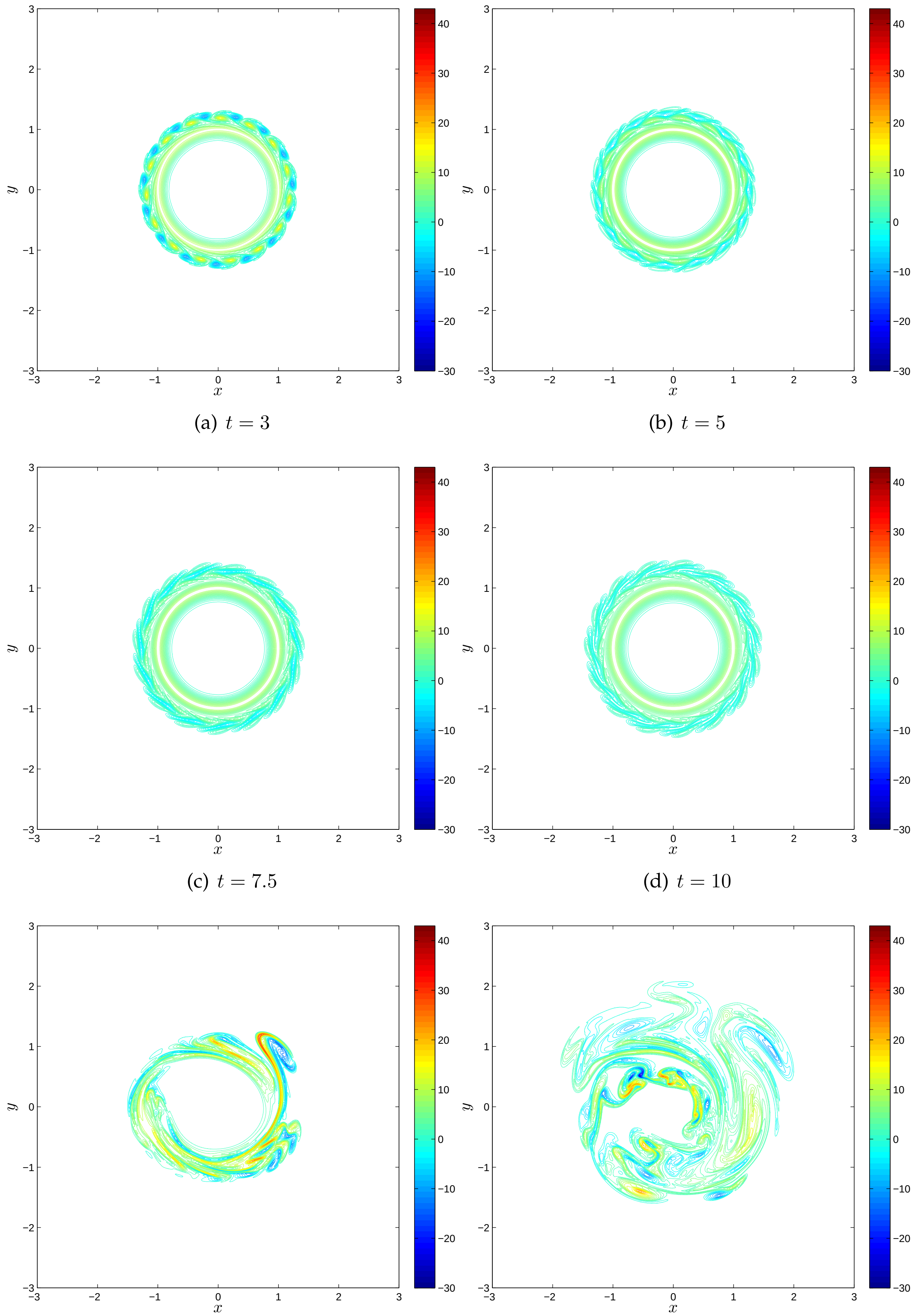


(e) $t = 2$



(f) $t = 2.5$

Figure A.7: CSI500: $Re = 5000, X_L = 0.5, N_r = N_{th} = 1024, R_{ext} = 6.0$

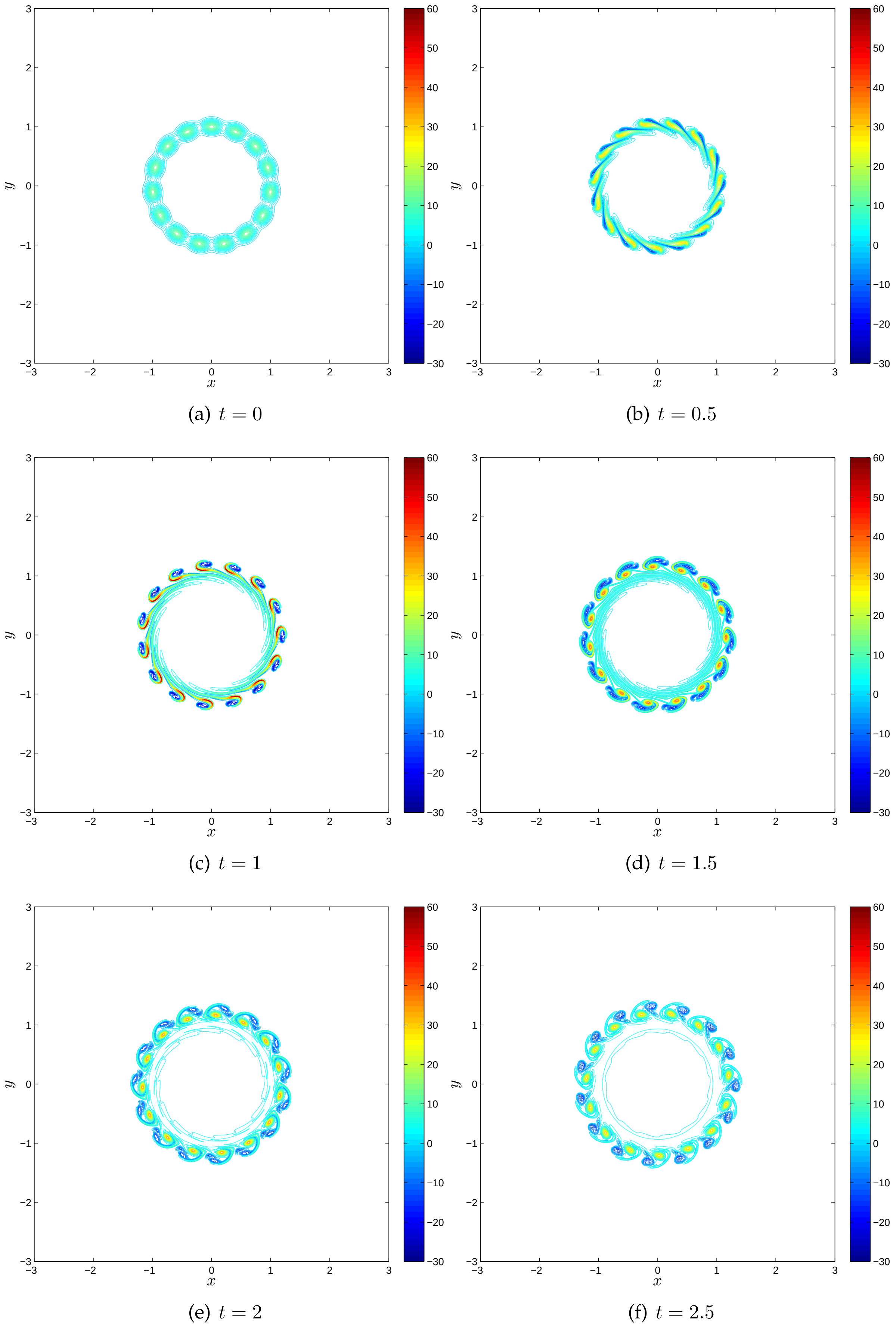


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

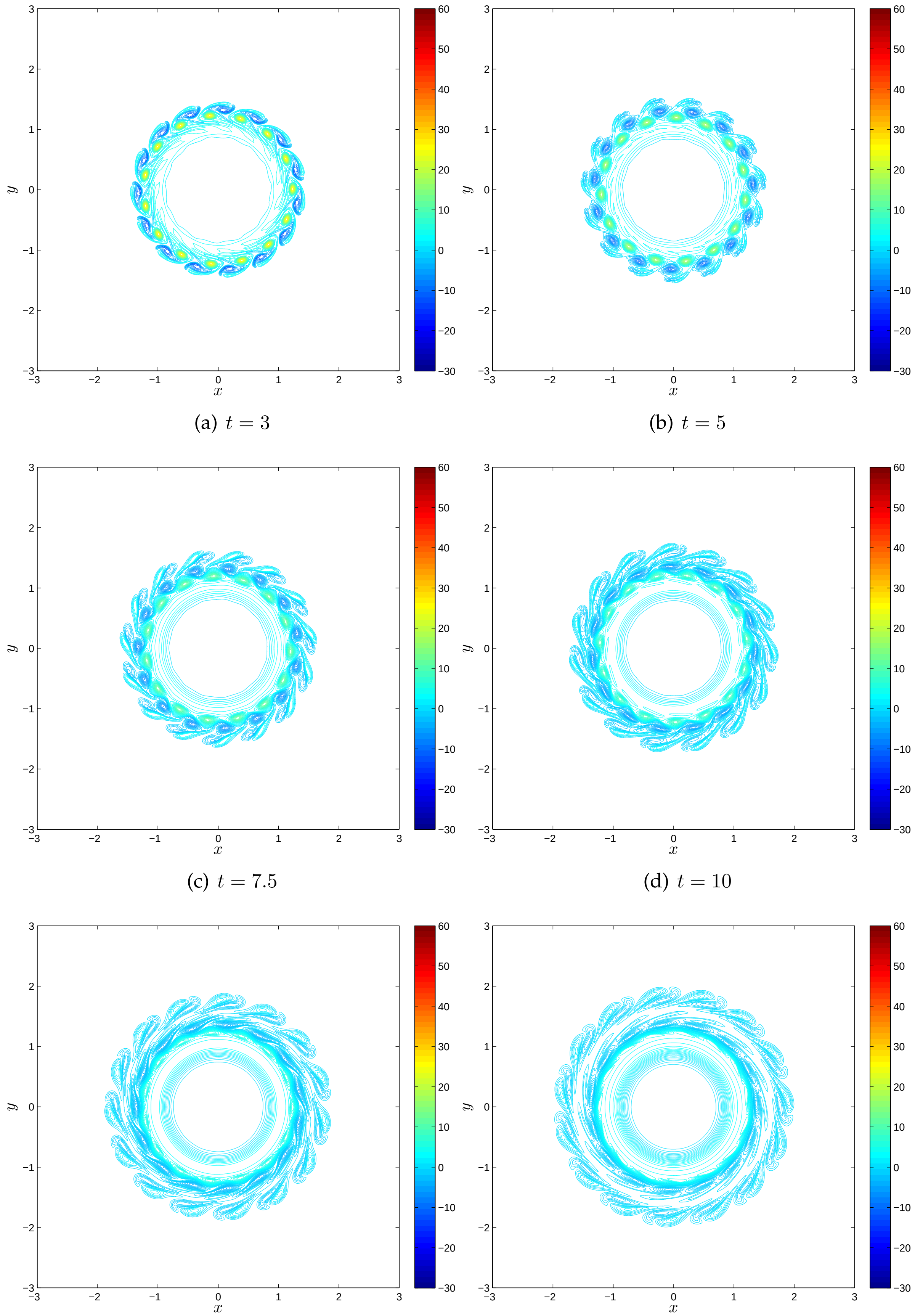
(e) $t = 15$ (f) $t = 20$

Figure A.8: CSI500: $Re = 5000$, $X_L = 0.5$, $N_r = N_{th} = 1024$, $R_{ext} = 6.0$



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 1.5$ (e) $t = 2$ (f) $t = 2.5$

Figure A.9: CSI600: $Re = 5000, X_L = 1, N_r = N_{th} = 1024, R_{ext} = 6.0$

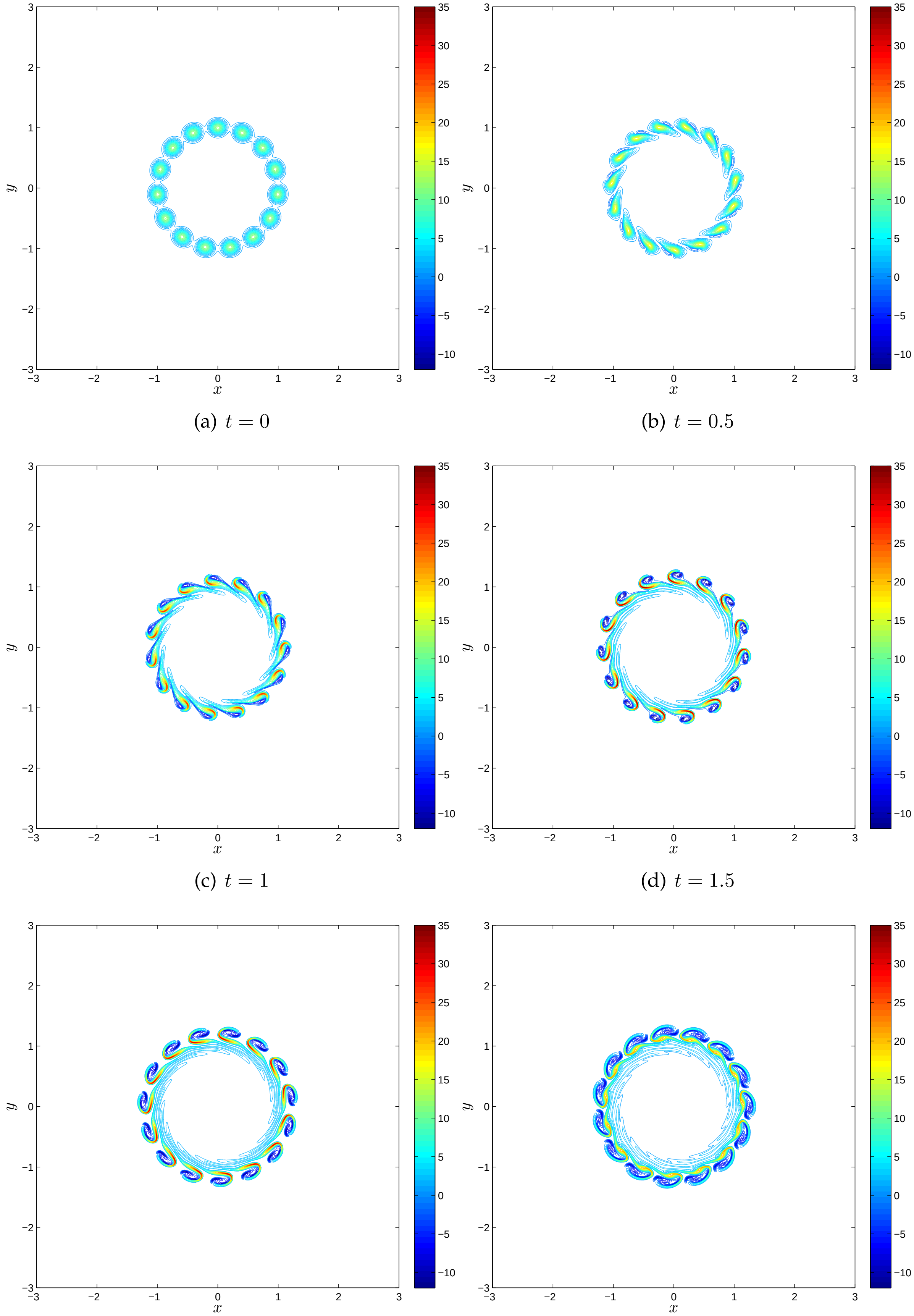


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.10: CSI600: $Re = 5000, X_L = 1, N_r = N_{th} = 1024, R_{ext} = 6.0$

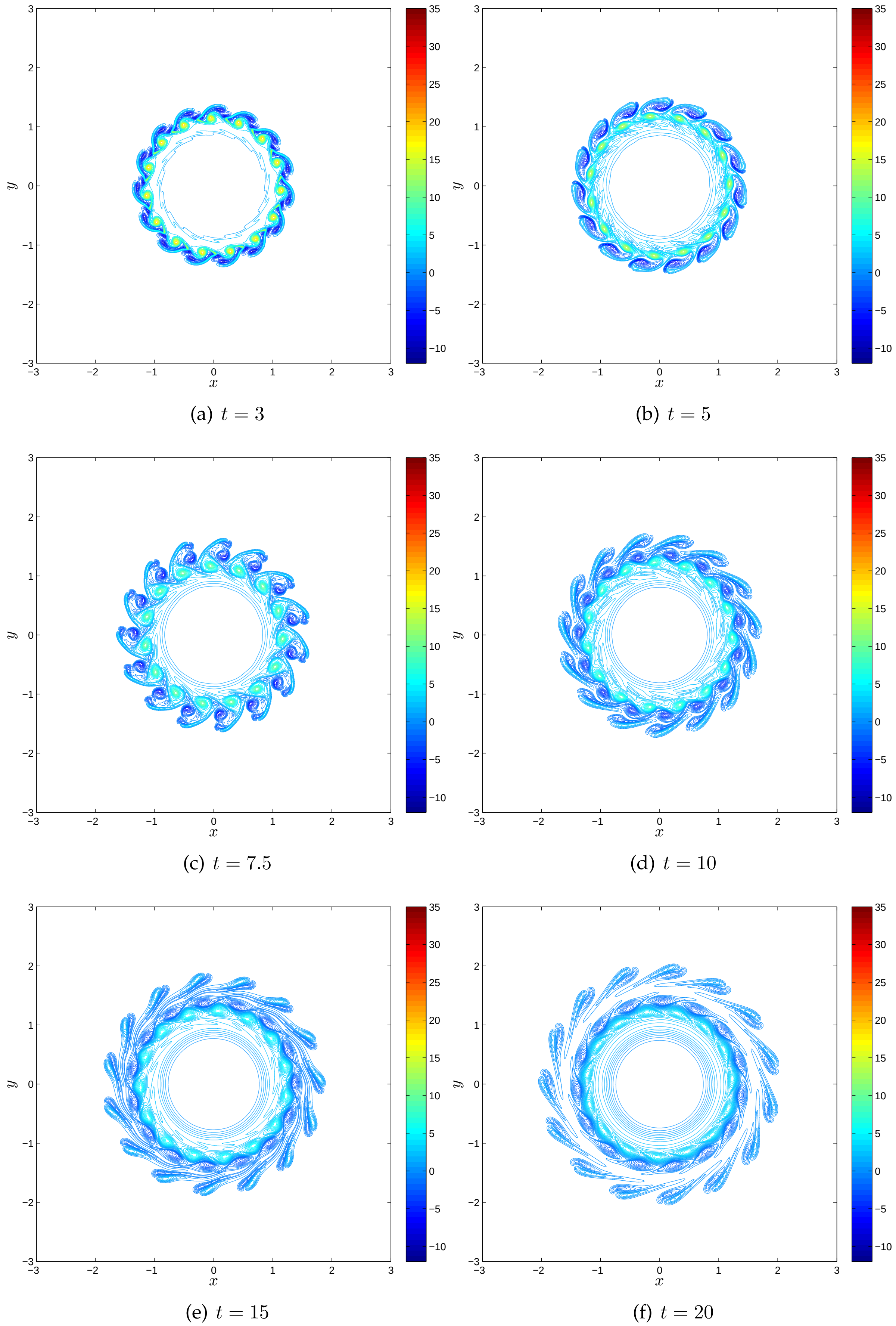


(a) $t = 0$ (b) $t = 0.5$

(c) $t = 1$ (d) $t = 1.5$

(e) $t = 2$ (f) $t = 2.5$

Figure A.11: CSI700: $Re = 5000, X_L = 2, N_r = N_{th} = 1024, R_{ext} = 6.0$

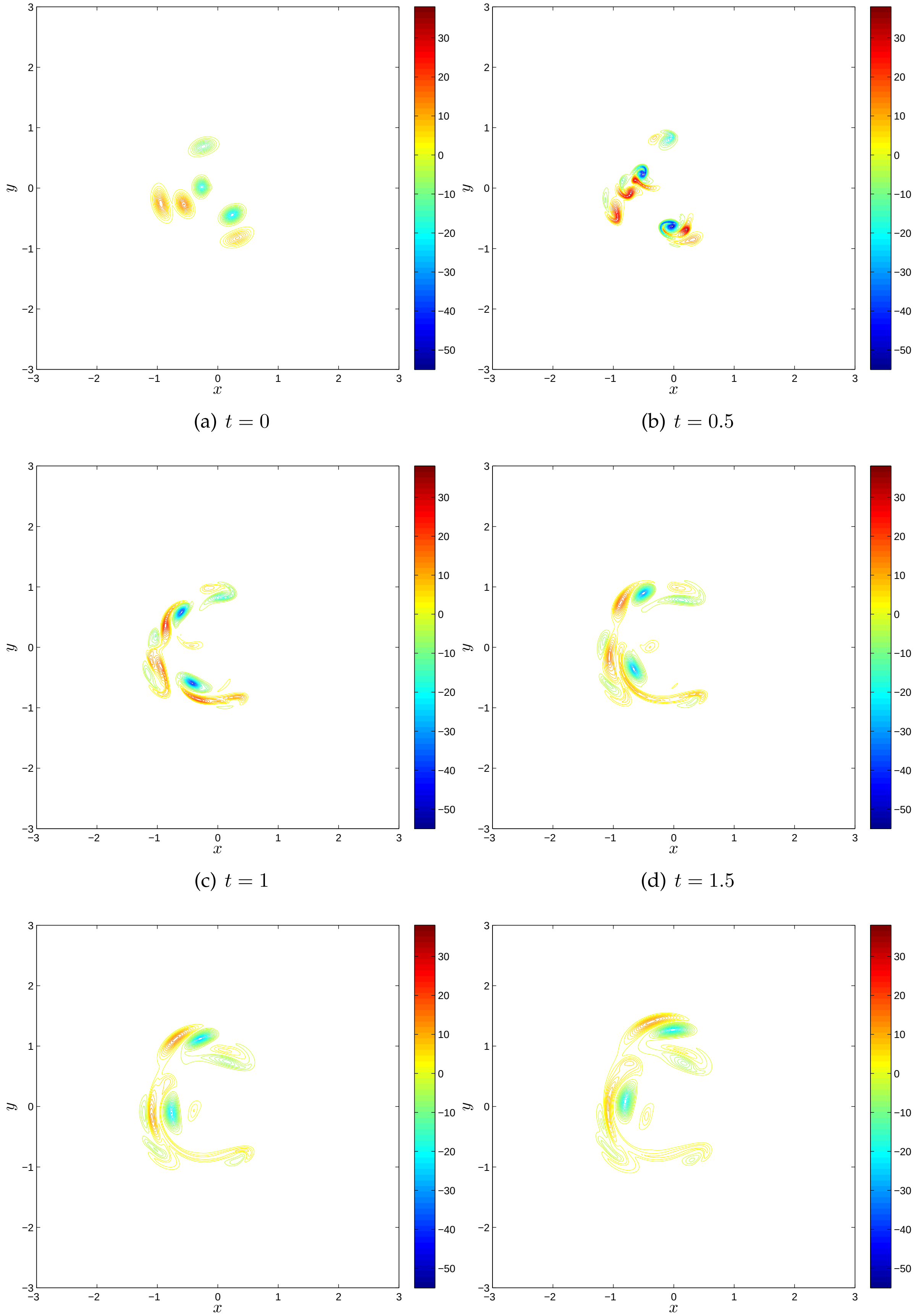


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.12: CSI700: $Re = 5000, X_L = 2, N_r = N_{th} = 1024, R_{ext} = 6.0$

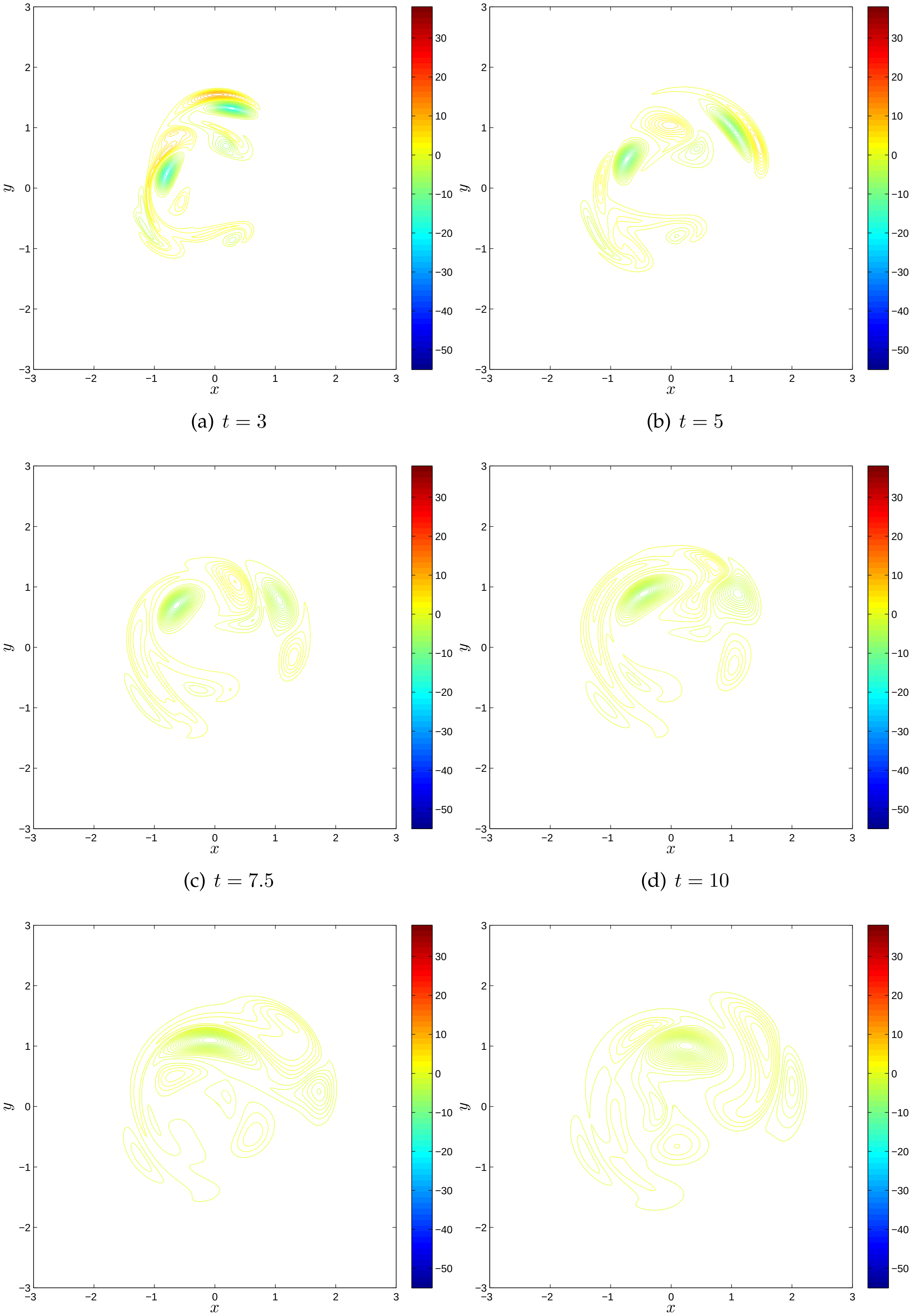


(a) $t = 0$ (b) $t = 0.5$

(c) $t = 1$ (d) $t = 1.5$

(e) $t = 2$ (f) $t = 2.5$

Figure A.13: C1: $Re = 1000, X_L = 0.5, N_r = N_{th} = 372, R_{ext} = 6.0$

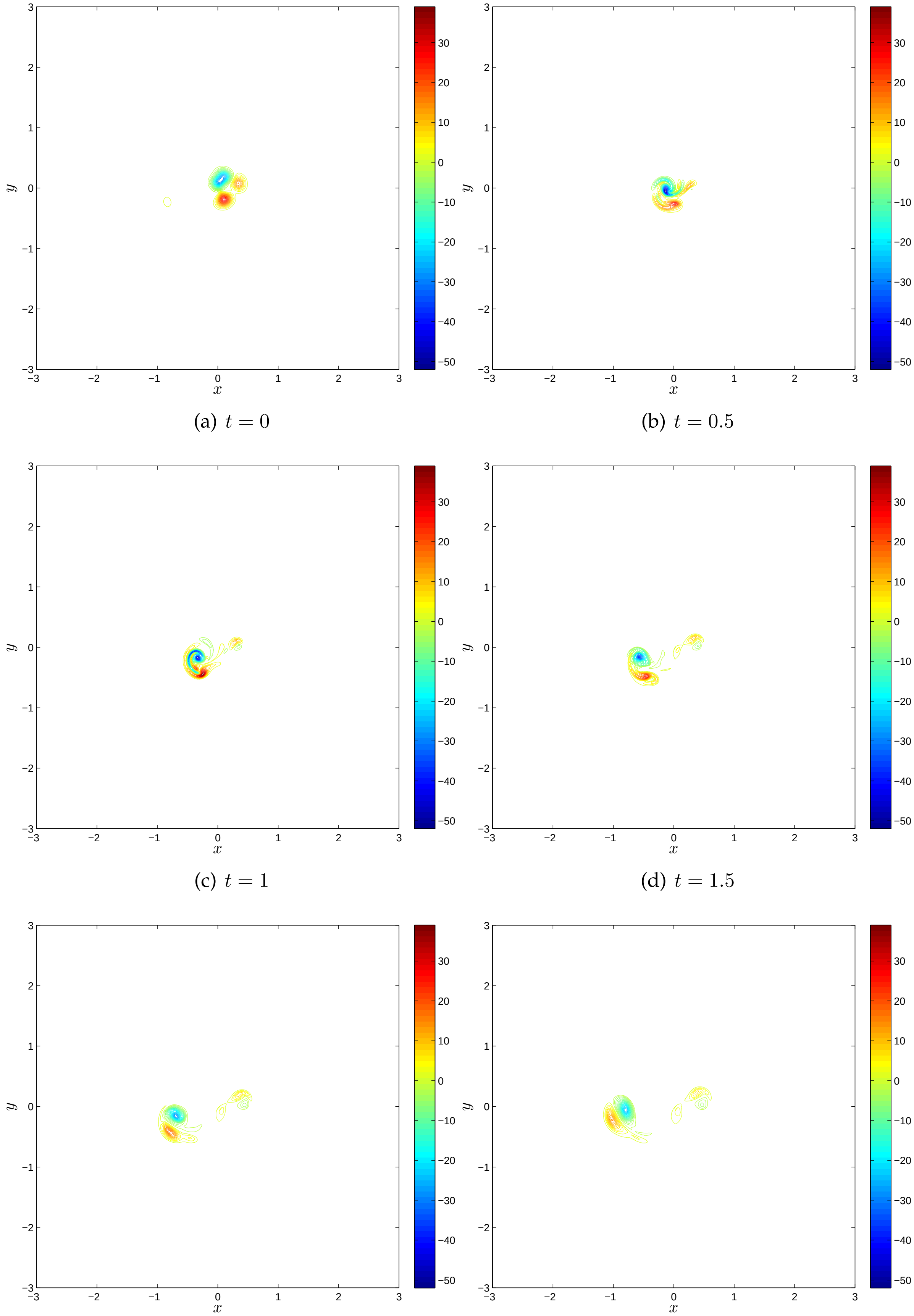


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.14: C1: $Re = 1000, X_L = 0.5, N_r = N_{th} = 372, R_{ext} = 6.0$

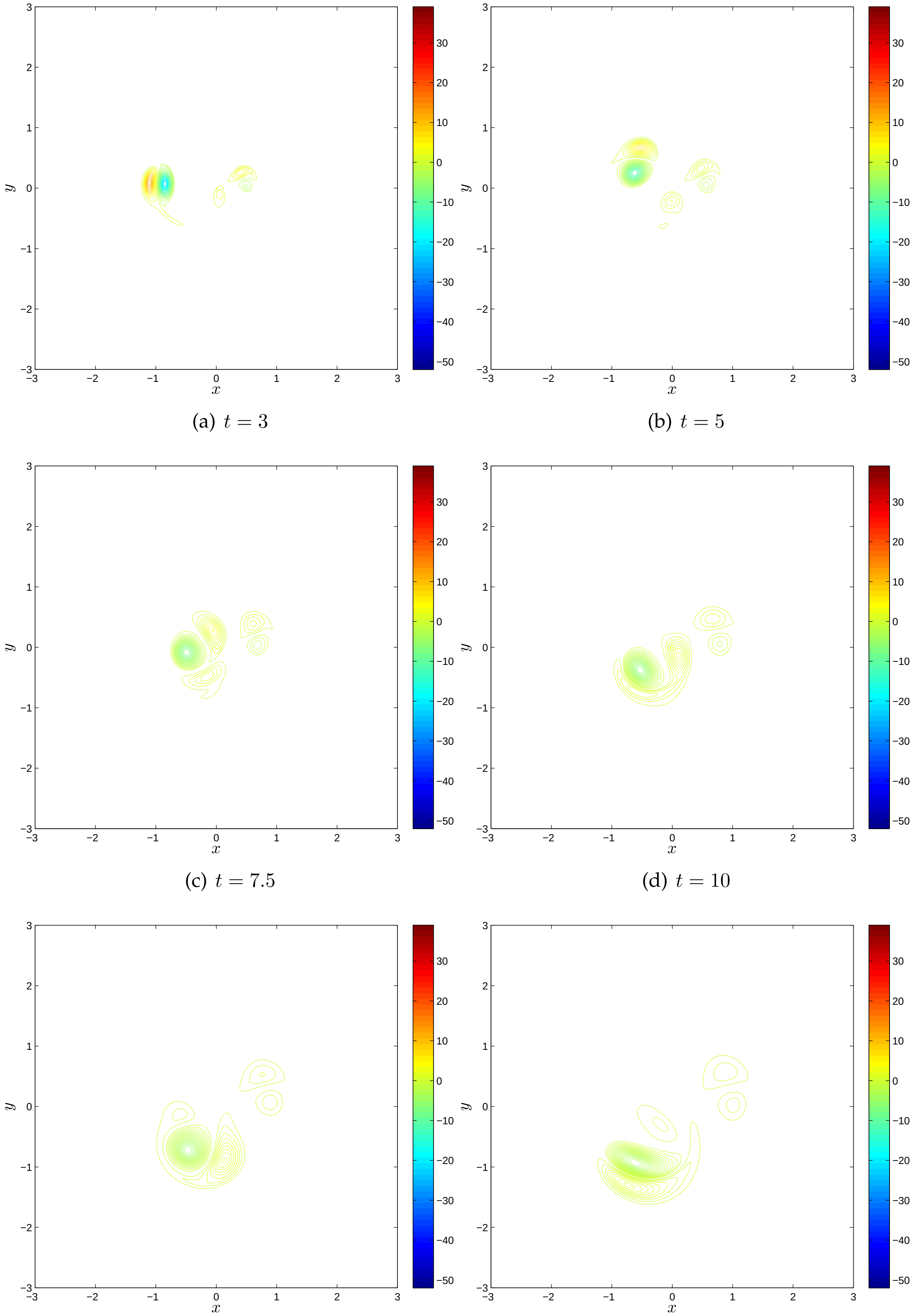


(a) $t = 0$ (b) $t = 0.5$

(c) $t = 1$ (d) $t = 1.5$

(e) $t = 2$ (f) $t = 2.5$

Figure A.15: C2: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$

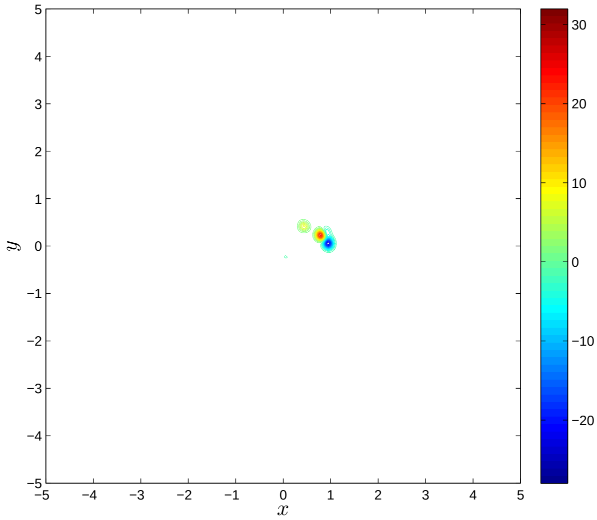


(a) $t = 3$ (b) $t = 5$

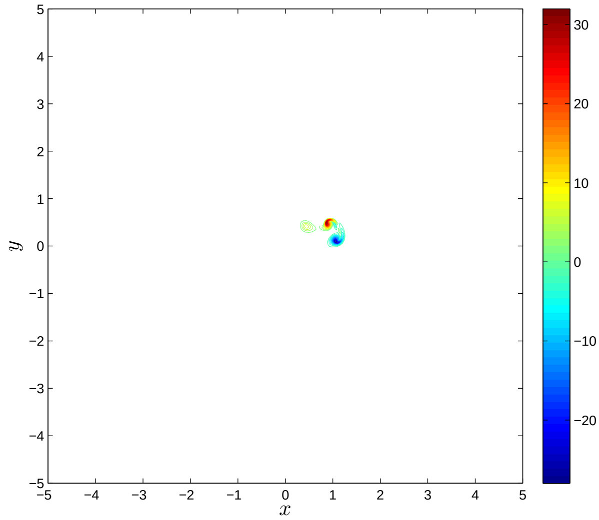
(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

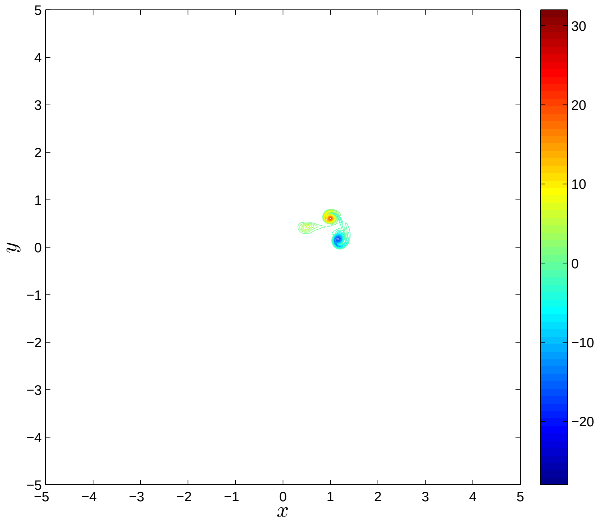
Figure A.16: C2: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$



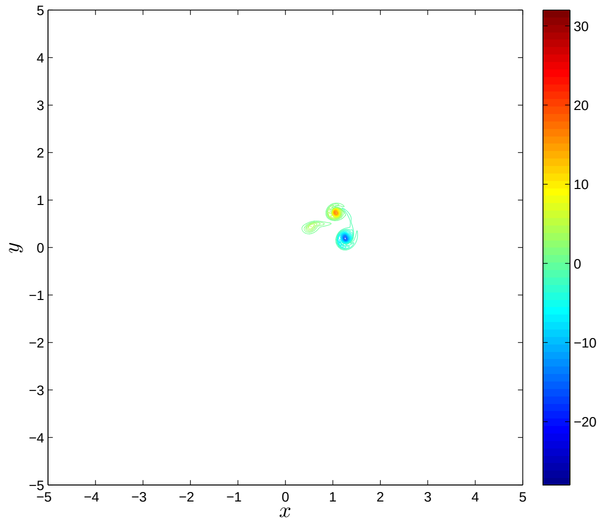
(a) $t = 0$



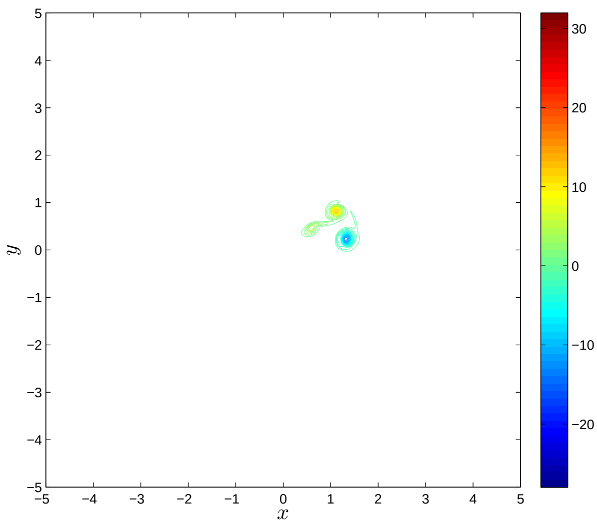
(b) $t = 0.5$



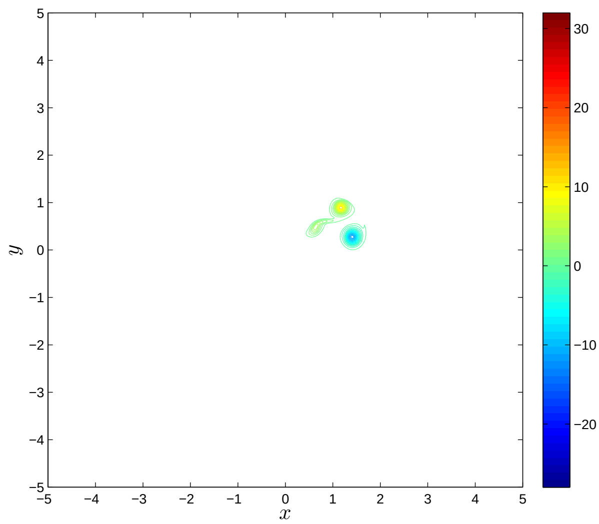
(c) $t = 1$



(d) $t = 1.5$

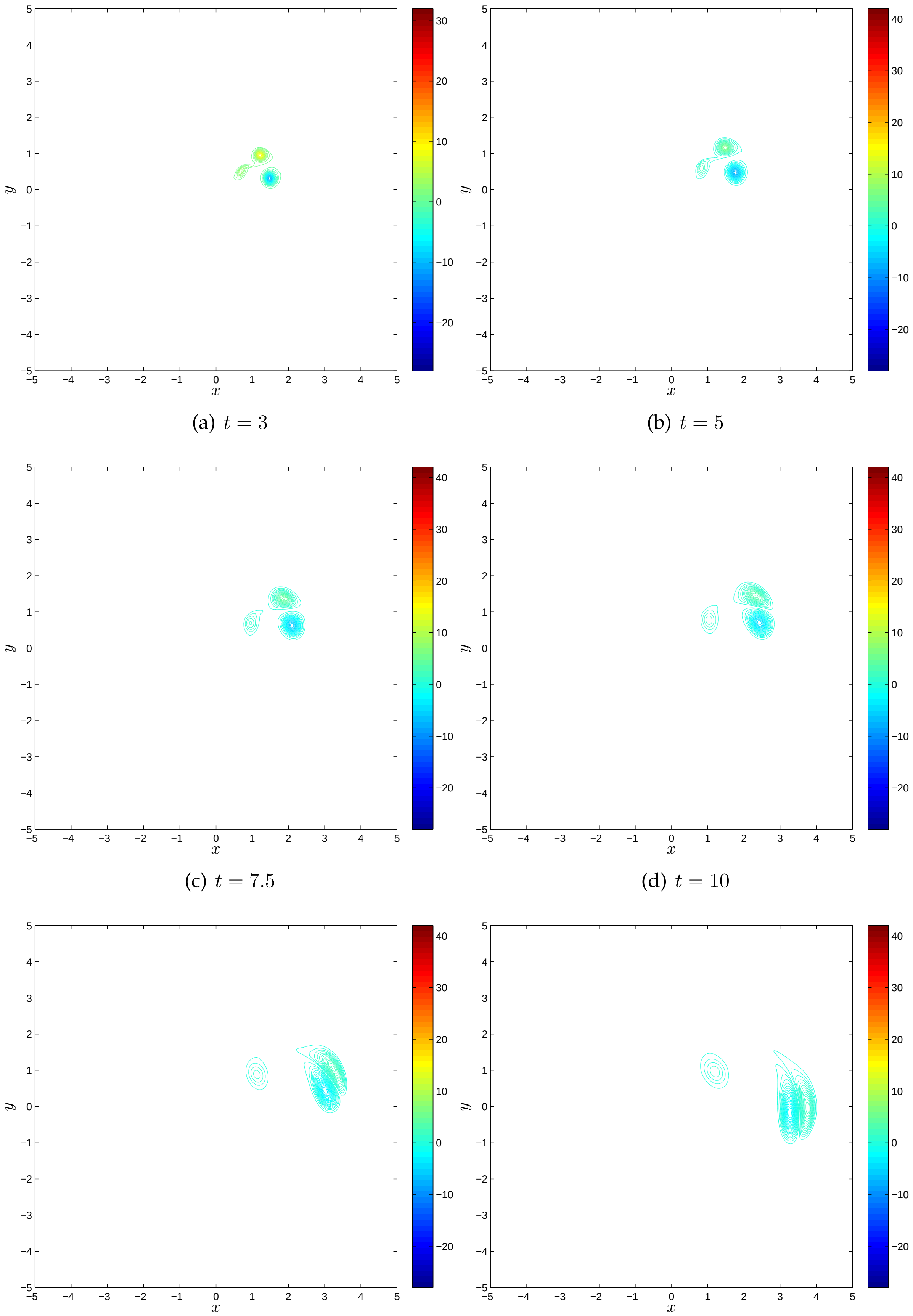


(e) $t = 2$



(f) $t = 2.5$

Figure A.17: C3: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$

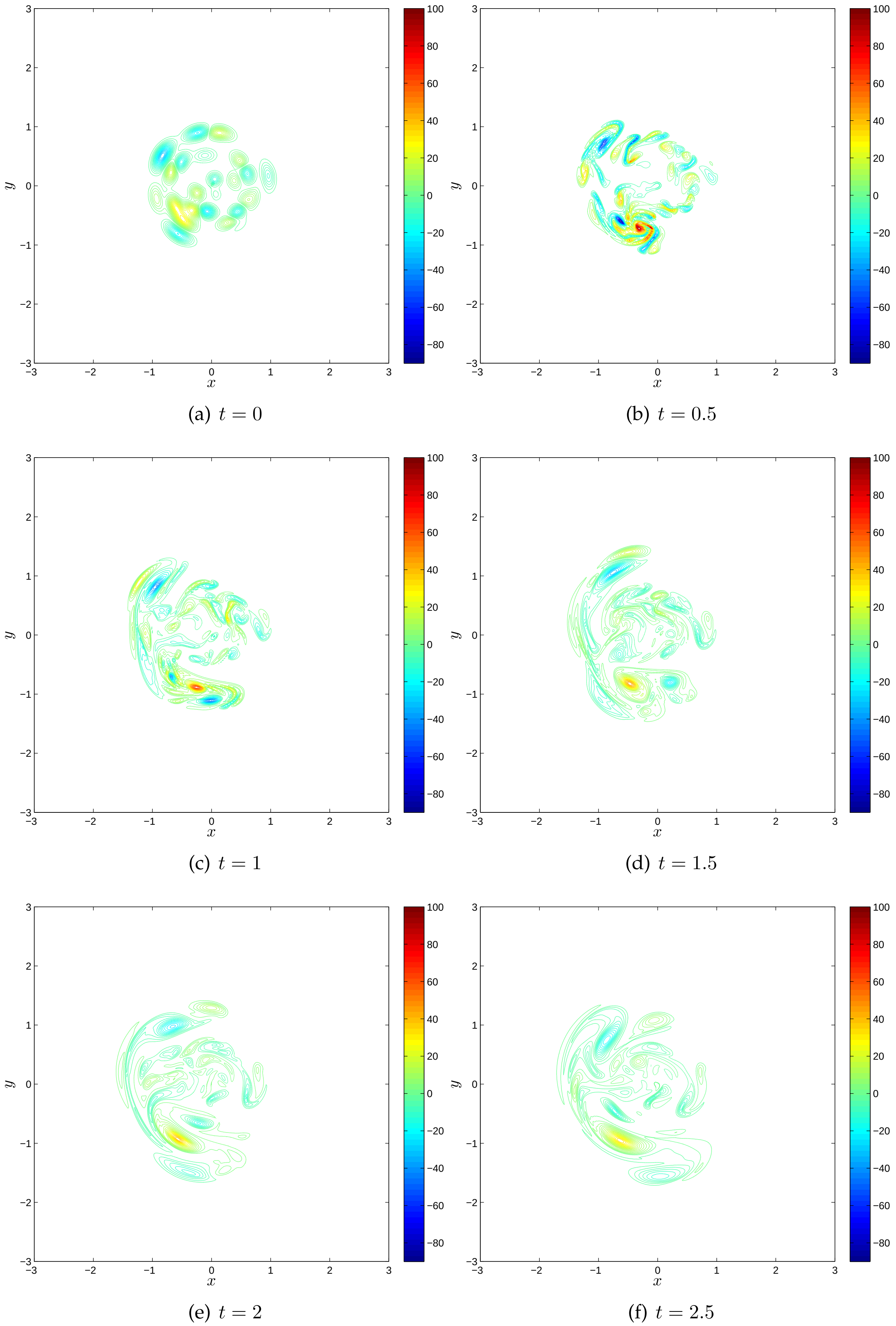


(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.18: C3: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$



(a) $t = 0$

(b) $t = 0.5$

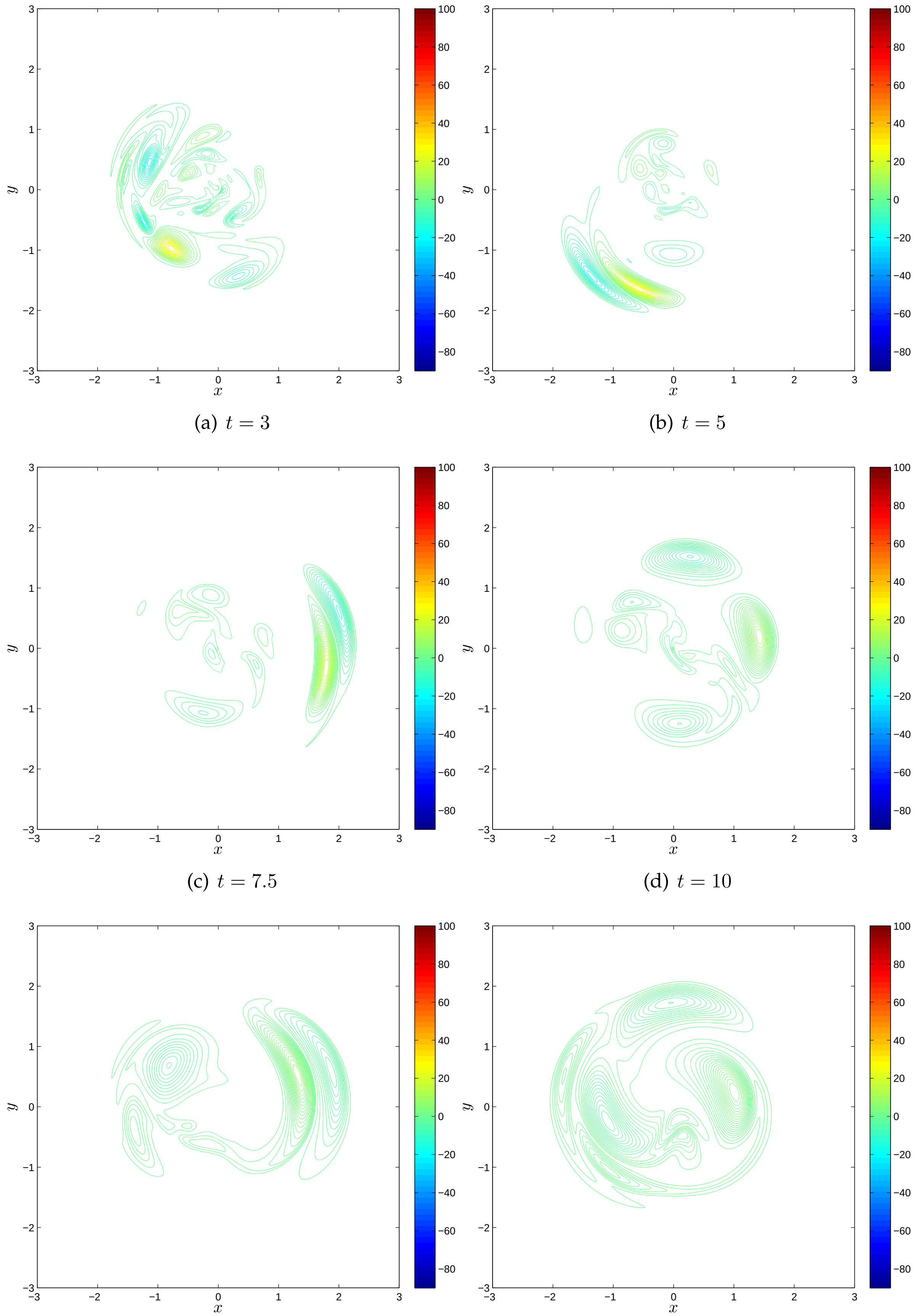
(c) $t = 1$

(d) $t = 1.5$

(e) $t = 2$

(f) $t = 2.5$

Figure A.19: C100: $Re = 1000, X_L = 0.5, N_r = N_{th} = 372, R_{ext} = 6.0$



(a) $t = 3$ (b) $t = 5$

(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

Figure A.20: C100: $Re = 1000$, $X_L = 0.5$, $N_r = N_{th} = 372$, $R_{ext} = 6.0$

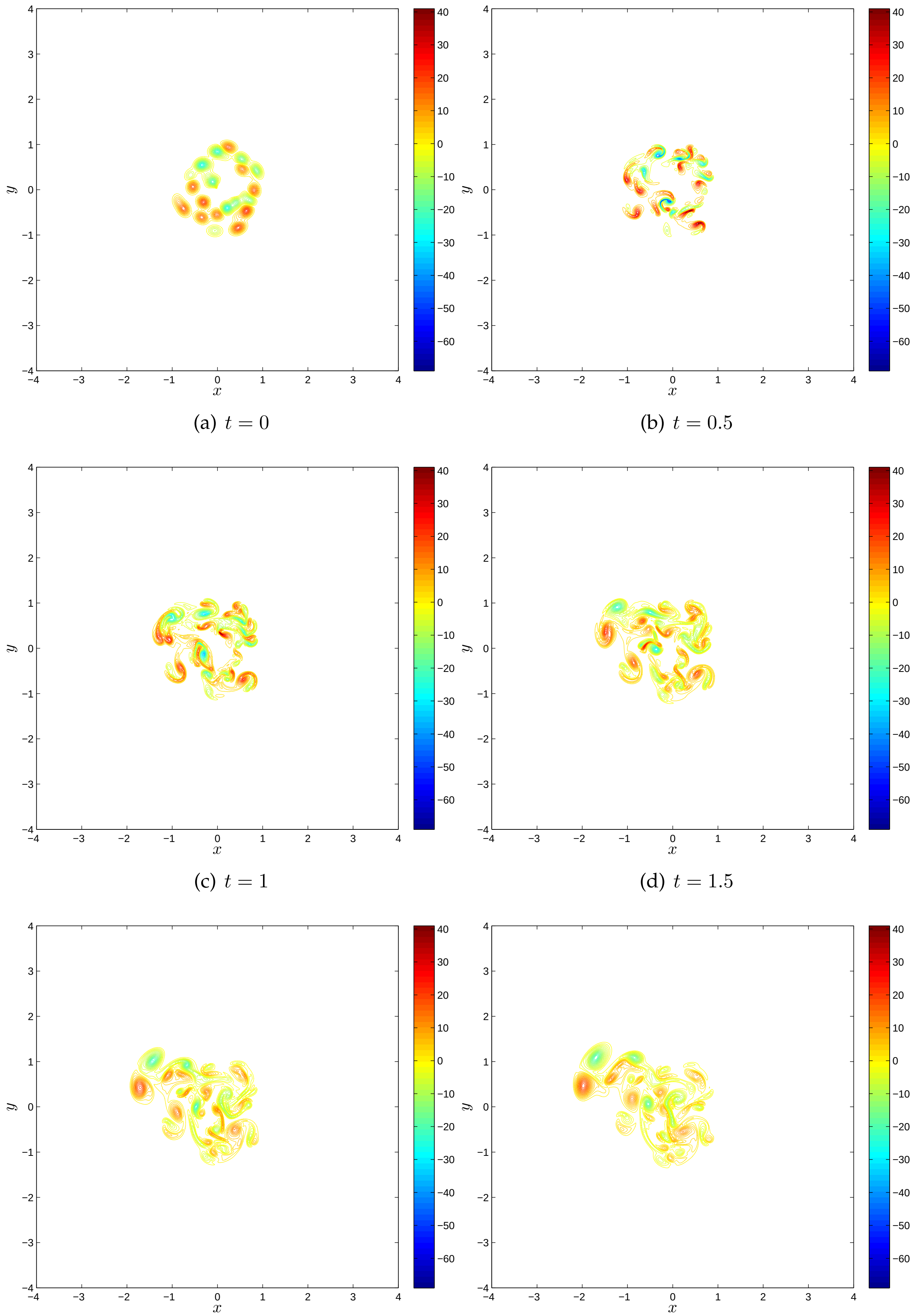
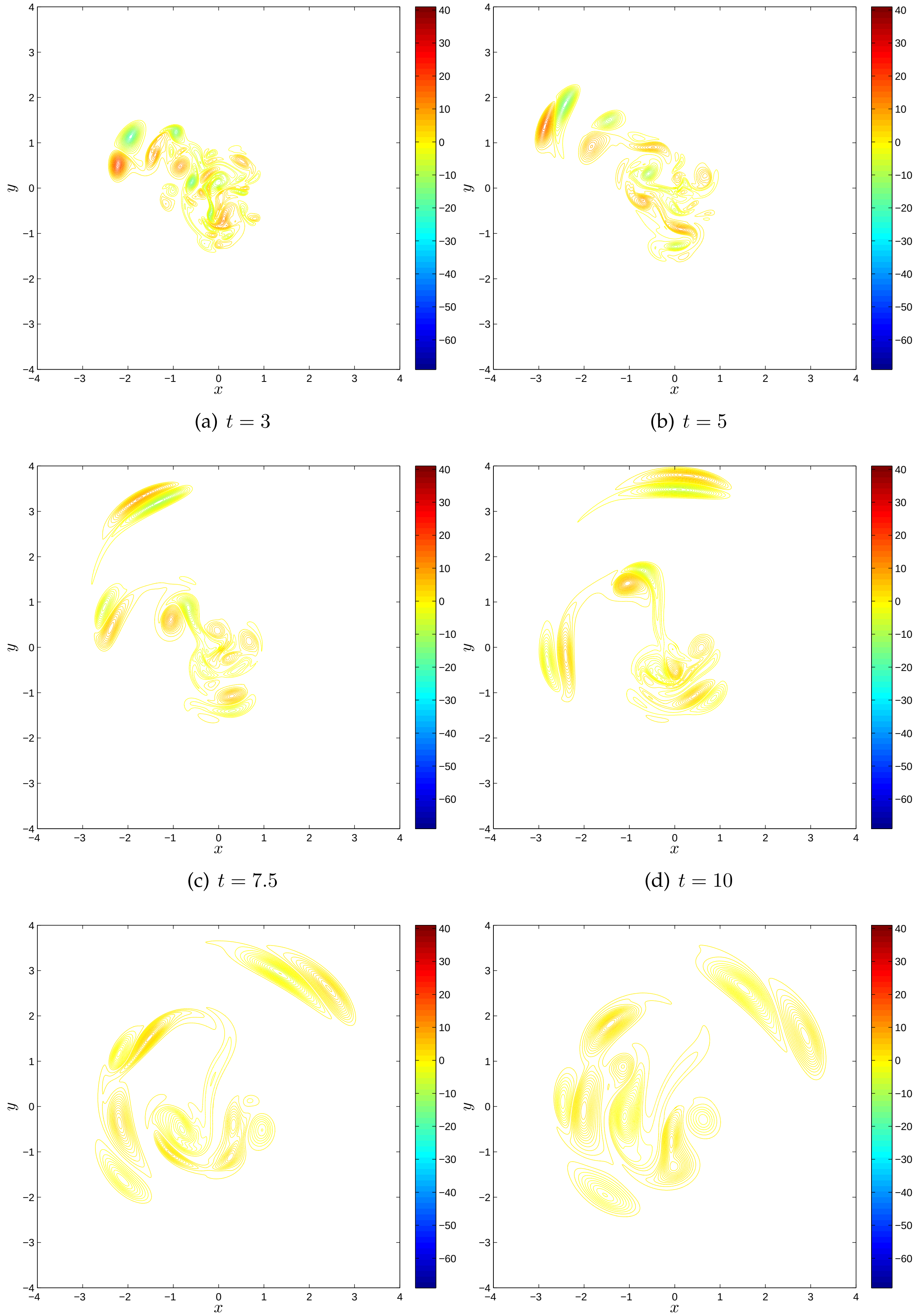


Figure A.21: C200: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$

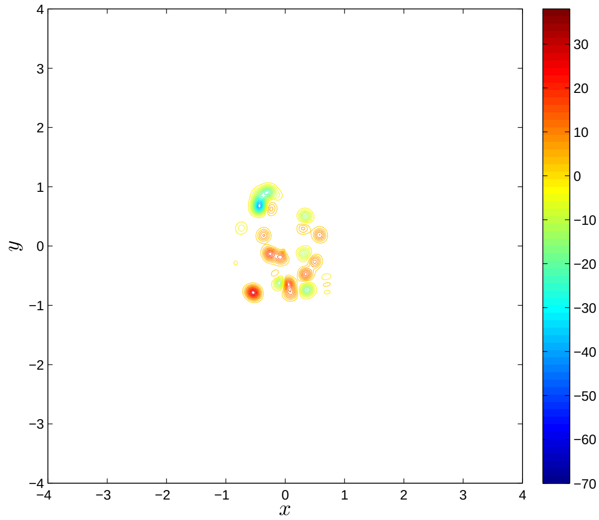


(a) $t = 3$ (b) $t = 5$

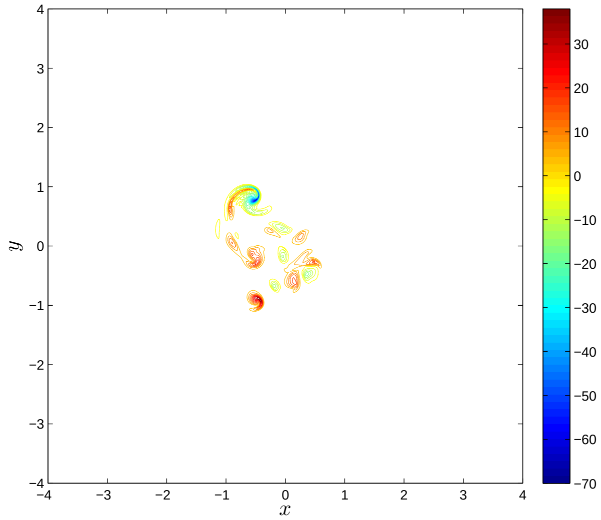
(c) $t = 7.5$ (d) $t = 10$

(e) $t = 15$ (f) $t = 20$

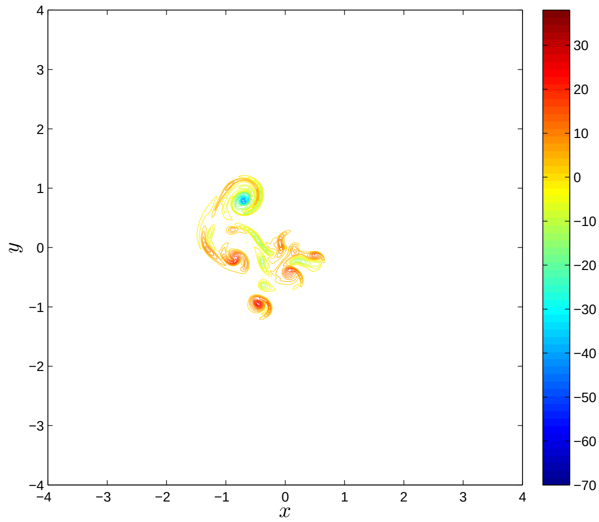
Figure A.22: C200: $Re = 1000, X_L = 1, N_r = N_{th} = 372, R_{ext} = 6.0$



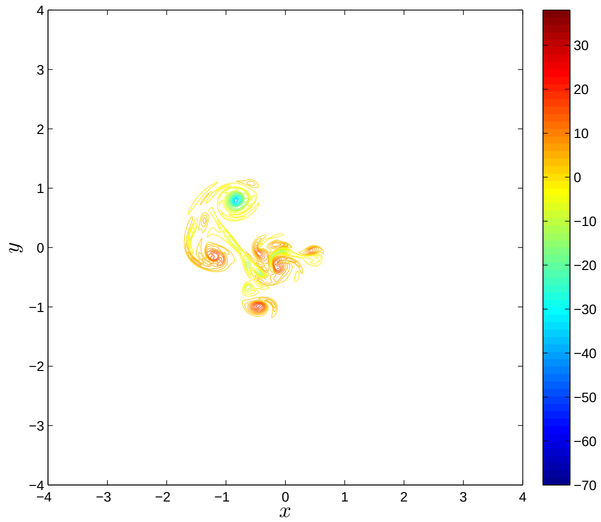
(a) $t = 0$



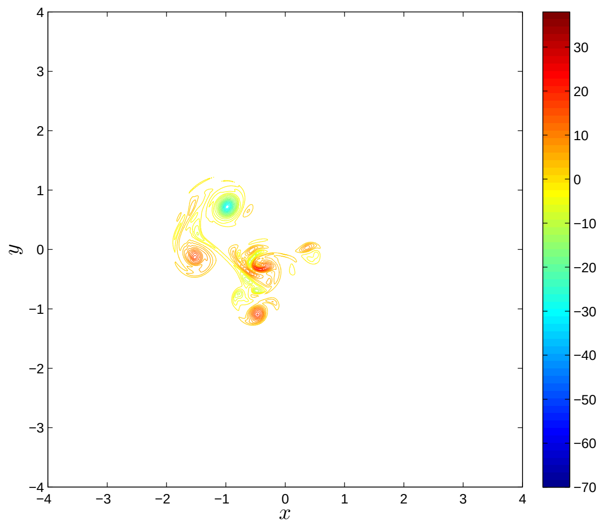
(b) $t = 0.5$



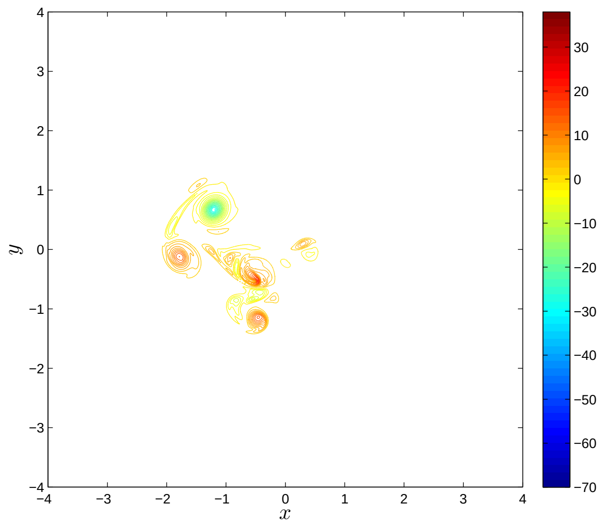
(c) $t = 1$



(d) $t = 1.5$



(e) $t = 2$



(f) $t = 2.5$

Figure A.23: C300: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$

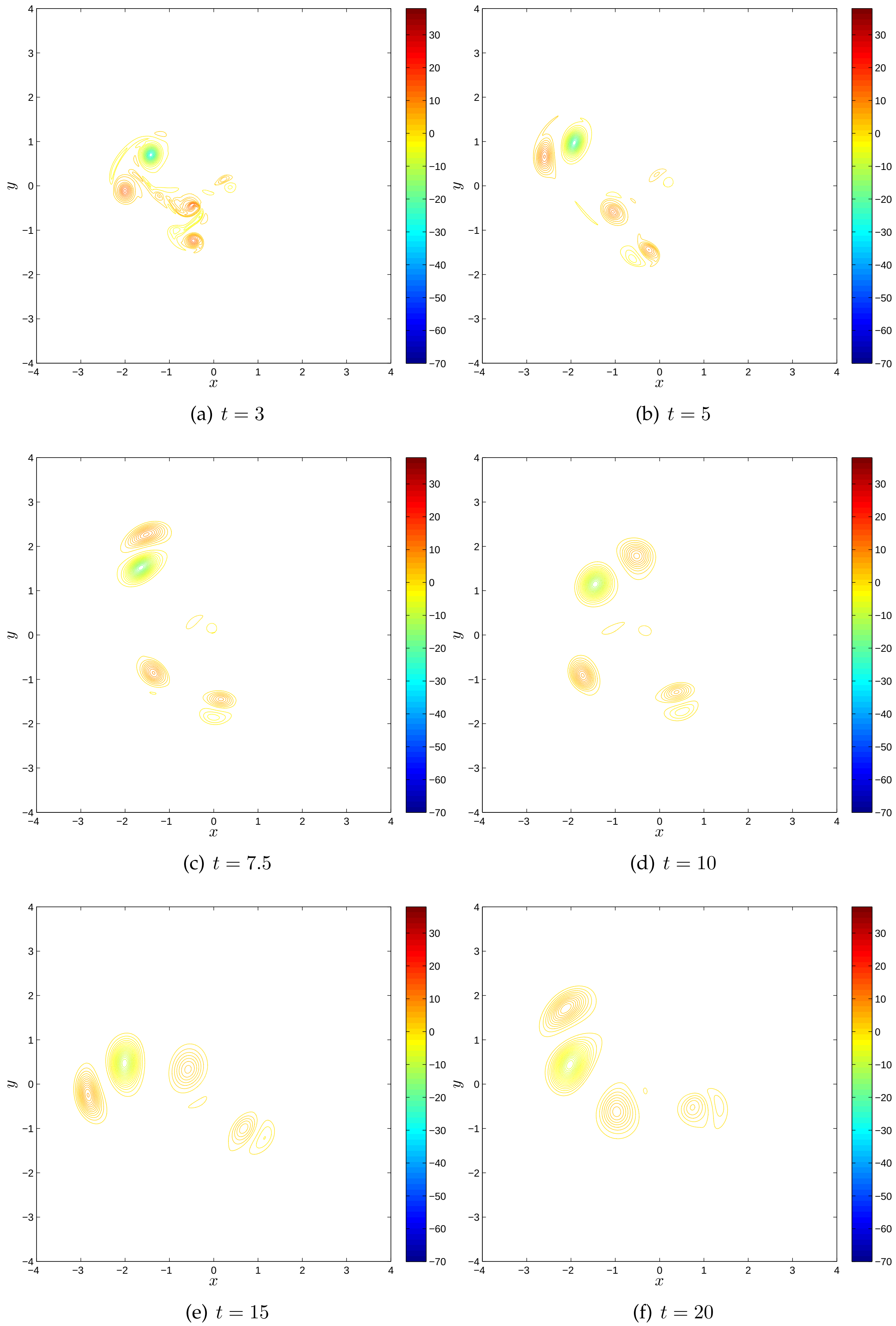


Figure A.24: C300: $Re = 1000, X_L = 2, N_r = N_{th} = 372, R_{ext} = 6.0$

Lebenslauf

Olga Kelbin

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19/09 - 22/09/10	interdisciplinary Turbulence initiative (iTi 2010), Bertinoro, Italy
21/06 - 24/06/11	vortices and waves: identifications and mutual influences (EUROMECH colloquium 531), Moscow, Russia
26/03 - 30/03/12	83rd annual meeting of the international association of applied mathematics and mechanics (GAMM), Darmstadt, Germany

09/09 - 13/09/12	9th European fluid mechanics conference (EFMC 9), Rom, Italy
30/09 - 04/10/12	interdisciplinary Turbulence initiative (iTi 2012), Bertinoro, Italy
01/09 - 04/09/13	14th European turbulence conference (ETC 14), Lyon, France